New Class of Exact Solutions to Einstein-Maxwell-dilaton Theory on Four-dimensional Bianchi Type IX Geometry

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Abstract

We construct new classes of cosmological solution to the five dimensional Einstein-Maxwell-dilaton theory, that are non-stationary and almost conformally regular everywhere. The base geometry for the solutions is the four-dimensional Bianchi type IX geometry. In the theory, the dilaton field is coupled to both the electromagnetic field and the cosmological constant term, with two different coupling constants. We consider all possible solutions with different values of the coupling constants, where the cosmological constant takes any positive, negative or zero values. In the ansatzes for the metric, dilaton and electromagnetic fields, we consider dependence on time and two spatial directions. We also consider a special case of the Bianchi type IX geometry, in which the geometry reduces to that of Eguchi-Hanson type II geometry and find a more general solution to the theory.

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EMD	Einstein-Maxwell-dilaton
CBR	Cosmic Background Radiation
FRW	Friedman-Walker-Robertson

1 Introduction

Gravity has been always a mysterious phenomena. There have been different insights about the gravity through the history, from its formulation as a force, acting in the distance between two massive bodies, which was presented by Newton, to its formulation in general relativity. In 20th century, Einstein formulated the general relativity, where the gravity was no more considered as a force. Gravity manifests itself as the curvature of the spacetime in general relativity. The latest theories dealing with gravity, mainly consider higher dimensional physics to emphasis the role of gravity. Constructing the exact solutions to the gravitational physics makes us one step closer to the understanding of the universe.

One of the main aims of gravitational physics, is to find the exact solutions to the Einstein gravity in the presence of matter fields in different dimensions. The ideas of higher dimensional gravity and dimensional compactification are covered in different areas of research [1, 2, 3]. Moreover, one can obtain a better insight about the holographic dual of a gravitational theory, by generalizing the solutions in asymptotically flat spacetime to de-Sitter and Anti-de-Sitter solutions [4]. The solutions to the Einstein theory in the background of different matter fields such as the Maxwell field, dilaton field and NUT charges are explored in [5, 6, 7]. The relevant solutions to the Einstein gravity in the presence of matter fields can be found in the compactification of M-theory in generalized Freund-Rubin theory [8]. The applications and properties of the Einstein-Maxwell-dilaton theory are covered in different areas, such as slowly rotation black holes [9, 10], cosmic censorship [11], gravitational radiation [12] and hyperscaling violation [13]. We explore the exact solutions to the five-dimensional Einstein-Maxwell-dilaton theory with two coupling constants and a cosmological constant.

We find the exact solutions to the Einstein-Maxwell-dilaton theory based on the Bianchi type IX metric. The dilaton field is coupled to the electromagnetic field and the cosmological constant with two different coupling constants. We consider the five-dimensional spacetime and generate the c-function and discuss the properties of the spacetime. We verify the ansatzes that we consider for the five-dimensional spacetime, Maxwell field and dilaton field satisfy all the field equations, by finding a relation between the coupling constants. We also consider the coupling constants to be equal to each other and find the exact solutions to the Einstein-Maxwell-dilaton theory for two different cases, where the coupling constant are equal to each other and non-zero, and where the coupling constants are equal to zero. Each case needs different ansatzes for the metric functions, Maxwell field and the dilaton field. Moreover, we present a combination of the solutions based on the four-dimensional Eguchi-Hanson space for three different cases of the coupling constants. The Eguchi-Hanson metric is a subspace of the Bianchi type IX geometry. We study both the Eguchi-Hanson and Bianchi type IX geometries and discuss their singularities and properties. We verify the ansatzes that we assume for the combined solutions for the Eguchi-Hanson space and show that our assumptions satisfy all the Einstein, Maxwell and dilaton field equations. Moreover, we find the cosmological constant in terms of the coupling constant and show that it can be positive, negative or zero. In the end, we discuss the uplifting of the found solutions to the higher dimensional theories such as Einstein gravity in higher dimensions and Einstein-Maxwell theory with a cosmological constant. We show that our solutions cannot be derived from the compactification of these theories. Moreover, we calculate the Kretschmann invariant and discuss the singularities of the spacetime.

In this thesis, there are five chapters discussing the theory of Einstein-Maxwell-dilaton, and the theoretical and mathematical background of the related theories. In chapter two, we present the Newton's theory of gravity, to the Einstein's theory of general relativity. We discuss the origin of the general relativity and mention the important equations without deriving them. We introduce the Lorentz transformations and show how different Lorentz inertial frames are related to each other. Moreover, we study the important mathematical objects that are essential for constructing general relativity. We show how the curvature of the spacetime is related to the energy-momentum tensor by introducing the Einstein equations. Understanding this theory requires a good background in differential geometry since the main idea of gravity is presented in terms of the curvature tensor and its deviations. Moreover, due to the non-linearity of the equations, finding the exact solutions to this theory is essential. Finding the exact solutions requires imposing extra constraints and specific conditions. As an example, we mention the Schwarzschild's solution to the general relativity and discuss its metric. We find the geodesics of the Schwarzschild metric and find the trajectories for both massive particles and mass-less particles, such as photons. We also present the Kerr and Reissner-Nordstrom solutions, as two other exact solutions to the theory of general relativity. We finish the chapter by introducing the Friedmann-Robertson-Walker model in cosmology and study its metric.

We use the four-dimensional Bianchi type IX as the background geometry to find the exact solutions to the five-dimensional Einstein-Maxwell-dilaton theory. We dedicate chapter three to the mathematical properties of the Bianchi geometry. For a better understating, we include all the other types of Bianchi geometry and discuss their properties, their applications and their important sub-spaces. The Bianchi type spaces are all included in the classification of the homogeneous spaces by Bianchi. We follow Bianchi's procedure and represent the classification explicitly. Homogeneity is an important condition that one considers for the spacetime. As an example, in the Friedmann-Robertson-Walker model in cosmology, the spacetime is considered to be homogeneous and isotropic. Moreover, after introducing the Bianchi classification, we study the Bianchi type IX geometry in detail. We show that the Bianchi type IX metric satisfies all the Einstein field equations. We also study a few important sub-spaces of the Bianchi type IX metric, which have many applications in different areas of research. We introduce the Atiyah-Hitchin metric as a sub-space of the Bianchi geometry and study its limits, which leads to the Taub-NUT metric. Moreover, we explain the Jacobi elliptic functions and their properties and show the behaviour of these functions in different figures (as these functions are being used in constructing the above metrics). Finally, we introduce the triaxial Bianchi type IX geometry and study its properties as well as the Eguchi-Hanson type I and type II geometries, which appear in the limits of the Bianchi type IX metric.

Our goal in chapter four is to construct and study the Einstein-Maxwell-dilaton action. Hence, we start the chapter by introducing the Lagrangian, the action and their relation in classical field theory. After explaining the Hamilton's principle of least action, we derive the Euler-Lagrange equations, which indicate the equations of motion for a system. As an example, we find the equation of motion for a real massless scalar field. We continue the chapter by introducing a proper Lagrangian for the electromagnetic field in the presence and absence of the electromagnetic source to construct the Maxwell's action. After deriving the equations of motion for Maxwell's field, we construct the Einstein-Hilbert action in the vacuum as well as in the presence of the matter. As the scalar dilaton field appears from the compactification of higher dimensions in different theories, we discuss Kaluza's idea as the first theory that includes higher dimensional gravity. Finally, we reach our goal by constructing the Einstein-Maxwell-dilaton action, where the dilaton field is coupled to both the electromagnetic field and the cosmological constant with two different coupling constants. We study two well-known solutions to the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II geometry and Taub-NUT geometry. We explain the considered ansatzes for the five-dimensional metric, electromagnetic gauge field and dilaton field and find a relation between the coupling constants. Moreover, we find the cosmological constant in terms of the coupling constant and show that it can be negative, positive or zero. We explain the solutions in different cases where the coupling constants are non-zero and not equal to each other, and where they are non-zero and equal to each other.

In chapter five we find a new class of cosmological solutions to the Einstein-Maxwell-dilaton theory, where the dilaton field is coupled to the electromagnetic field as well as the cosmological constant, with two different coupling constants. The geometry that we use as the background is the Bianchi type IX geometry. We construct the action for the Einstein-Maxwell-dilaton field in five dimensions and derive the Einstein field equations by varying the action with respect to the metric tensor. Moreover, by varying the action with respect to the electromagnetic gauge field and the dilaton field, we find the Maxwell and dilaton field equations in five dimensions, respectively. We consider ansatzes for the five-dimensional line element, the electromagnetic gauge field and the dilaton field. By calculating the Maxwell and Einstein field equations, we determine the constants that appear in the ansatzes. Moreover, we find the cosmological constant and show that it takes positive, negative or zero values. We also find a relation between the coupling constants. There are three different cases to this theory. Each case is discussed separately and the solutions are given for each of them. The first case is where the coupling constants are non-zero and not equal to each other, the second case is where the coupling constants are non-zero and equal to each other, and the third case is where they are both zero. Each case needs different ansatzes. A few constraints will be found during the calculation and we will show that under some circumstances, our results lead to the well-known theories. We also discuss the uplifting of the found solutions to the higher dimensional theories such as Einstein gravity in higher dimensions, and the Einstein-Maxwell theory with a cosmological constant and we show that our solutions cannot be found from the compactification of these theories. Moreover, we calculate the Kretschmann invariant and discuss the singularities of the spacetime. There are also graphs showing the behaviour of the metric functions, electric field and dilaton field. C-theorem and its correspondence with the contraction/expansion of the spacetime are also discussed in this chapter and also found for our results. The exact solutions to the Einstein-Maxwell-dilaton theory are well-known based on the Eguchi-Hanson type II geometry. We find new combined solutions to this theory based on the Eguchi-Hanson type II geometry in terms of our solutions and show that these solutions satisfy all the equations.

In the end, we study the Janis-Newman method for generating rotational solutions from the static one. Janis and Newman found the Kerr metric by applying their method on the Schwarzschild metric. We also represent an alternative way that was proposed by Giampieri and show how this method works by finding the Kerr-Newman metric from the Reissner-Nordstrom metric. We sum up the results and also mention our further research in the conclusion.

2 General Relativity

The concept of gravity has been always an important issue for scientists. There have been a lot of attempts for formalizing gravity with different notions behind it for centuries. One of the most successful attempts was done by Isaac Newton. Classical mechanics was introduced by Newton and Leibniz in the 17th century with the powerful mathematical tool, named differential calculus. In this theory, gravity was introduced as a force between matters (with mass) that act in the distance. Until the 20th century, physics was divided into three main parts namely mechanics, which was the study of matters and their dynamics with different forces (including gravity), electromagnetic theory, which concentrates on the wave-like phenomena such as radiation, and thermodynamics, which was about the interaction between matters and waves [14]. Meanwhile, some phenomena could not be justified by the knowledge of that time. Quantum mechanics and general relativity were needed in order to explain those strange phenomena. Our focus is on general relativity, a powerful theory which was introduced by Einstein in the 20th century. In this theory, gravity is not considered as a force anymore. It is a feature of the geometry of the spacetime itself. In this theory, every physical quantity is described as a geometrical entity and laws of physics are geometrically expressed as relationships between these geometrical objects [15]. For gaining a better understanding of the general theory of relativity, two things are required: familiarity with the Newtonian gravity and the knowledge of the differential geometry for dealing with the geometrical and algebraic quantities.

2.1 Newtonian Gravity

In Newtonian mechanics, gravity is introduced as a force between two massive objects that acts in the distance. This force depends on the masses of the matters and is inversely proportional to the square of the distance between them. This relationship can be formulated by the gravitational constant as below [16]:

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{e}_r, \tag{2.1}$$

where m_1 and m_2 represent the masses of the first and second object, r is the distance between the masses, $\hat{e}_r$ is the unit vector in r direction and G is the gravitational constant, which is $G = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$ in SI units. Also, we consider the origin of the coordinate system at the position of one of the masses.

An important feature of the classical mechanics is that the equations are invariant under the Galilean transformation. Assume an event P in an inertial frame S with orthogonal Cartesian axes x, y and z. Inertial frame in Newtonian mechanics is defined according to the first law of Newton, in a way that a free particle has zero or constant velocity in that frame of reference. Now consider another inertial frame S' that its coordinates coincide with the coordinate of the frame S at $t = t' = 0$ and has a constant velocity v in x direction with respect to S. The coordinates of the event P can be presented in S' frame with the Galilean transformation between these inertial frames as follow:

$$x' = x - vt, \tag{2.2}$$

$$y' = y, \tag{2.3}$$

$$z' = z, \tag{2.4}$$

$$t' = t, \tag{2.5}$$

where x', y' and z' are the spatial coordinates in three dimensional Euclidean space and t' is the time in S' frame [17]. An important principle in Newtonian mechanics is that there is a universal time, i.e., time is the same in all inertial frames, regardless of their motion. The geometry of the Newtonian mechanics is the flat three dimensional Euclidean space and the distance between two events can be calculated as [18]:

$$\Delta s = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}. \tag{2.6}$$

This distance is invariant under the Galilean transformation. In other words, the length of an object or the distance between two events in one frame is the same in all other inertial frames. It can be shown that the main equation of Newtonian mechanics is also invariant under this transformation, i.e., the equation takes the same form in different inertial frames. The second law of the Newtonian mechanics implies that:

$$\vec{F} = M\vec{a}, \tag{2.7}$$

where $\vec{F}$ is the force acting on the matter, M is the inertial mass of the matter and $\vec{a}$ is the acceleration of the matter due to the force $\vec{F}$. Consider the S and S' frames that were mentioned before. By applying the Galilean transformation, this law would be transferred as:

$$\vec{F'} = \vec{F}. \tag{2.8}$$

This equation implies that the laws of mechanics are invariant under the Galilean transformation.

Although all the classical mechanics equations are invariant under the Galilean transformation, scientists realized that the equations of electrodynamics are not the same in different inertial frames. i.e., Maxwell's equations are not invariant under the Galilean transformations. This was in contrast to the assumed postulate which states that every law of physics should take the same form in different inertial frames. The first attempt was to rewrite the electrodynamics theory in a way that it becomes invariant under the Galilean transformations. All the attempts failed. On the other hand, a Dutch physicist, Hendrik Lorentz, found a class of transformations that leaves the electromagnetic equations (Maxwell's equations) invariant. But it was Albert Einstein who realized the importance of Lorentz work and invented the theory of relativity.

2.2 From Special to General Relativity

Einstein considered changes in classical mechanics to make them invariant under the new transformations, which were called the "Lorentz Transformation". Einstein built his theory of special relativity based on an important principle, which implies that in every inertial frame, laws of physics take the same form. Moreover, he replaced the concept of universal time in classical mechanics with a new postulate which states that the speed of light is the same in all inertial frames [19, 20].

Consider an event P happening in a frame S. The coordinates of this event can be found in another frame, call it S', through Lorentz transformation. Note that S' is a frame that has a constant velocity v in x direction relative to S and at $t = t'$ the origins of these two frames coincide:

$$ct' = \gamma(ct - \beta x), \tag{2.9}$$

$$x' = \gamma(x - \beta ct), \tag{2.10}$$

$$y' = y, \tag{2.11}$$

$$z' = z, \tag{2.12}$$

where the primed parameters are the coordinates of event p in S'. Also $\beta = \frac{v}{c}$ (where c is the speed of light) and $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$. It can be seen from these transformations that time and space are meshed together. The old concept of independent space and time was replaced by the concept of spacetime as a four dimensional continuum. Maxwell's equations are all invariant under this transformation and moreover, the speed of light is the same in every inertial frame. This transformation can be represented in a more compact and useful way. We introduce Λ to represent the Lorentz transformations. For this case, we can introduce a second rank tensor $\Lambda^{\mu}\ _{\nu}$ in matrix representation as below:

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.13}$$

The geometry of the spacetime as a four dimensional manifold is called the Minkowskian geometry and the line element, which shows the distance between two events in a frame, is calculated as below:

$$ds^2 = (cdt)^2 - d\vec{r}^2, \tag{2.14}$$

where c is the speed of light and r is $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. It can be seen that this line element does not imply Euclidean geometry. The line element is invariant under the Lorentz transformation, meaning $ds' = ds$. This behaviour of the line element suggests that it should be a geometrical property of spacetime. This interval can be positive, negative or zero. Depending on the sign, the interval is called time like, space like and light like (or null) when it is positive, negative or zero, respectively:

If $ds^2 > 0$, the interval is called *time-like*,

If $ds^2 < 0$, the interval is called *space-like*,

If $ds^2 = 0$, the interval is called *light-like* or *null*.

The line element in equation (2.14) can be written in a more compact way:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.15}$$

The Greek indices run from 0 to 3 by considering $x_0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$. In this equation (2.15), $g_{\mu\nu}$ is a symmetric second-rank tensor, which is called the metric tensor. In some references, the metric of the Minkowskian spacetime is shown by $\eta_{\mu\nu}$. We represent this metric by a 4×4 matrix:

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.16}$$

A few amazing results of the special relativity would be the length contraction and the time dilation, which are respectively as below:

$$l = \frac{l_0}{\gamma}, \tag{2.17}$$

$$\tau = \tau_0 \gamma, \tag{2.18}$$

where γ is $\gamma = (1 - v^2/c^2)^{1/2}$. In the first equation, l_0 is the proper length of two events in frame S', measured in the same frame. An observer in S finds this length as l. Also, τ_0 is the proper time between two events in the S' frame and τ is the time that an observer in frame S measures for the same pair of events.

In special relativity, gravity is not considered. Einstein proposed the general relativity in 1915 by including gravity as a geometrical entity in the language of differential geometry. In Newtonian mechanics, gravity is considered as a force that acts on a particle with the gravitational mass m_G and gives the particle an acceleration with the inertial mass m_I of the particle [21]:

$$\vec{F} = G \frac{m_G M}{r^2} \hat{e_r} = m_G \vec{g} = m_I \vec{a}, \tag{2.19}$$

where $\vec{a}$ is the acceleration vector of the particle and $\vec{g}$ is the gravitational field, which has the following relation with the gravitational potential Φ [19]:

$$\vec{g} = -\vec{\nabla}\Phi, \tag{2.20}$$

where $\vec{\nabla}\Phi$ is the gradient of the gravitational potential.

The gravitational potential in Newtonian mechanics can be found from $\vec{\nabla}^2\Phi = 4\pi G\rho$, where G is the gravitational constant and ρ is the gravitational matter density. It can be seen from this equation that the gravitational potential in Newtonian gravity does not depend on time. In other words. changes in the matter distribution lead to the instantaneous changes in the gravitational potential. This is obviously in contrast with the postulate of special relativity, which considers the speed of light as the limit. Therefore, another theory with different principles was needed to describe gravity.

Einstein proposed the general theory of relativity by considering the equivalence principle, which states that in a small region of spacetime, the laws of physics in a freely falling frame are the same as special relativity. This idea was formed by the similarity of the behaviour of matters while they are in a uniform gravitational field and when they have acceleration due to a freely falling motion. In this theory, gravity manifests itself as the curvature of the spacetime. Hence, the spacetime geometry in global aspect is not flat anymore. Gravity is proposed as a behaviour of the geometry in the reaction of the matter-energy. In other words, the geometry of spacetime indicates the motion of the matter and matter indicates the geometry of spacetime [15]. In general relativity, physical objects are geometrical entities and laws of physics are explained as geometrical relations between these objects [22].

In the theory of relativity, the old notion of separate space and time was replaced by the spacetime as a continuum which is represented by manifolds in mathematics. An N-dimensional manifold is a set that can be continuously parameterized. N-dimension indicates that for specifying any point on the manifold, N independent parameters are needed [19]. As the spacetime is curved, there is no global Lorentz reference frame. Referring to the equivalence principle of general relativity, at the infinitesimal neighbourhood of any point, a local Lorentz frame exists that laws of physics are the same as special relativity.

In curved spacetime, one deals with curved geometry and the metric tensor $g_{\mu\nu}$ would not be equal to $\eta_{\mu\nu}$ in general (which represents the metric of the flat spacetime). The metric tensor can be calculated by the basis coordinate vectors as:

$$g_{\mu\nu} = e_\mu . e_\nu, \tag{2.21}$$

where e_μ and e_ν are the basis coordinate vectors. Depending on the spacetime, $g_{\mu\nu}$ differs. It is worth noting that writing down the equations with invariant mathematical objects is essential, as it ensures the invariance of the equations in different Lorentz frames. We can define a local Lorentz frame in the neighbourhood of any point by considering the metric tensor to be equal to the one in flat spacetime and also considering that the covariant derivative of the metric tensor vanishes in this local frame:

$$g_{\mu\nu} = \eta_{\mu\nu}, \tag{2.22}$$

$$\partial_\gamma g_{\mu\nu} = 0, \tag{2.23}$$

where $\eta_{\mu\nu}$ is given in equation (2.16).

Gravity can be defined explicitly as the deviation of two nearby geodesics in spacetime. The concept of geodesic and deviation are yet to be defined. Geodesic is the path that a particle follows in the free falling motion. In other words, a geodesic is a curve that when is measured in a local Lorentz frame in its path, it acts as a straight and uniformly parameterized line [21]. The parameter that defines the geodesic is called the affine parameter, which is linearly related to the particle's proper time for massive particles.

$$\lambda = a\tau + b, \tag{2.24}$$

where λ is the affine parameter, τ is the proper time and a and b are two constants.

As we mentioned, gravitation manifests itself as the spacetime curvature, which shows up in the deviation of two nearby geodesics. To find an equation for the deviation of two geodesics, we need to introduce more mathematical objects in curved spacetime.

2.3 Four-vectors and the Covariant Derivative

These mathematical objects can be transferred to each other in different Lorentz frames by Lorentz boost or rotation, which are all encoded in equation (2.13). A main geometrical object in curved spacetime is four-vector. Consider V as a four-vector, which is independent of any reference frame. We show this four-vector with its components as $V = V^{\mu}e_{\mu}$, where e_{μ} is the basis vector. This four-vector can be transferred to another Lorentz frames as below:

$$V'^{\mu} = \Lambda^{\mu}{}_{\nu}V^{\nu}, \tag{2.25}$$

where $\Lambda^{\mu}{}_{\nu}$ is given in equation (2.13). Moreover, tensors can also be transferred between the Lorentz frames. Consider a second rank tensor $T_{\mu\nu}$ defined in a Lorentz frame, S. This tensor transfers to another Lorentz frame, S', as below:

$$T'_{\mu\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}T_{\rho\sigma}. \tag{2.26}$$

Due to the curvature of the spacetime, by transferring the vector from one point to another point, the direction of the vector might change. For example, imagine a closed circle that a four-vector is defined at an arbitrary point on it. By transferring this vector around the circle and returning it to the initial point, the direction of the vector will no longer be the same. Hence, the derivation of a vector should be different than the usual derivation in the flat spacetime. The reason of this difference is that in curved geometry, the basis vector also changes through the transformation. Therefore, we introduce a new derivative as below:

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma^{\nu}{}_{\mu\lambda}V^{\lambda}. \tag{2.27}$$

In this equation (2.27), ∇_{μ} is called the *covariant derivative.* ∂_{μ} in this equation is the usual derivative in flat spacetime and $\Gamma^{\nu}{}_{\mu\lambda}$ is the Christoffel symbol. The existence of the Christoffel symbol is due to this fact that the derivation of the basis vector is not zero in curved spacetime. Hence, we define the Christoffel symbol as:

$$\Gamma^{\mu}{}_{\nu\lambda} = e^{\mu}\partial_{\lambda}e_{\nu}, \tag{2.28}$$

where e^{μ} and e_{ν} are the contravariant and covariant vector bases, respectively [23]. Moreover, the Christoffel symbol is related to the metric tensor of the spacetime as:

$$\Gamma^{\mu}{}_{\nu\gamma} = \frac{1}{2}g^{\mu\sigma}(\partial_{\nu}g_{\sigma\gamma} + \partial_{\gamma}g_{\nu\sigma} - \partial_{\sigma}g_{\nu\gamma}). \tag{2.29}$$

The Christoffel symbol (2.29) is not a tensor, since it does not transform between the Lorentz frames as tensors do.

The covariant derivation in equation (2.27) can be generalized to tensors. As an example, consider a second rank tensor $T^{\mu}{}_{\nu}$. The covariant derivation of this tensor in curved geometry is:

$$\nabla_{\lambda} T^{\mu}{}_{\nu} = \partial_{\lambda} T^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\sigma\lambda} T^{\sigma}{}_{\nu} - \Gamma^{\sigma}{}_{\nu\lambda} T^{\mu}{}_{\sigma}. \tag{2.30}$$

Another important concept in differential geometry is the tangent space. At every point of a manifold, a tangent space can be define in a way that the tangent vectors at that point lie in that tangent space [24]. Although in the tangent space the applied laws on the vectors are the same as the flat geometry, another tool for comparing the tangent vectors at different points of a manifold is needed. This mathematical object is essential in general relativity, as it is used to compare the separation of geodesics, which manifests itself as gravity.

2.4 Geodesic, Parallel Transportation and the Curvature Tensor

In order to compare vectors in curved spacetime, we need to move them in a specific way by means of the parallel-transport. Consider a curve, parameterized by $x^{\alpha}(\lambda)$ in a manifold and v a vector at a point along this curve. The vector v can be transported along this curve as [19]:

$$\frac{Dv}{D\lambda} = \left(\frac{dv^{\mu}}{d\lambda} + \Gamma^{\mu}{}_{\nu\gamma} v^{\nu} \frac{dx^{\gamma}}{d\lambda}\right) = 0, \tag{2.31}$$

where λ is an affine parameter. This transportation of the vector v is called the parallel-transport along the curve.

We can give a better definition for the geodesic in terms of the tangent vector. Geodesic is a curve that its tangent vector is being parallel-transported along itself. We can formulate the geodesic equation as (more details on the derivation of the geodesic equation can be found in[22]):

$$\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}{}_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} = 0. \tag{2.32}$$

Solving this equation, one can find the geodesic equation of a particle in curved spacetime.

A practical approach for reaching to the geodesic equation is the Lagrangian method. The Euler-Lagrange equations enable one to generate the geodesic equations and also find the Christoffel symbols. We consider the following Lagrangian for an affinely parameterized geodesic $x^{\alpha}(\lambda)$ [19]:

$$L = g_{\alpha\beta} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda}, \tag{2.33}$$

where λ is an affine parameter. By substituting this Lagrangian (2.33) into the Euler-Lagrange equations, we find the geodesic equation:

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma} \dot{x}^{\beta} \dot{x}^{\gamma} = 0, \tag{2.34}$$

where “ $\dot{}$ ” denotes the derivation with respect to the affine parameter λ.

As it was mentioned, gravity shows itself in the deviation of two nearby geodesics. Consider two geodesics x^μ and x'^μ which are separated from each other by a vector ζ^α as $x'^\mu = x^\mu + \zeta^\mu$. The geodesic deviation of the mentioned geodesics x and x' are:

$$\frac{D^2\zeta^\mu}{D\sigma^2} + R^\mu{}_{\nu\rho\gamma}\zeta^\rho \frac{dx^\nu}{d\sigma}\frac{dx^\gamma}{d\sigma} = 0, \tag{2.35}$$

where $R^\mu{}_{\nu\rho\gamma}$ is called the *curvature tensor* or *Riemann tensor*. The curvature tensor is a rank four tensor that measures the curvature of the spacetime. It can be seen that the curvature tensor is being defined as the relative acceleration of the two nearby geodesics. The geodesic deviation implies that in a curved spacetime, the two geodesics that are initially parallel would either converge or diverge. The curvature tensor in terms of the Christoffel symbol is written as:

$$R^\alpha{}_{\beta\sigma\lambda} = \partial_\sigma \Gamma^\alpha{}_{\beta\lambda} - \partial_\lambda \Gamma^\alpha{}_{\beta\sigma} + \Gamma^\gamma{}_{\beta\lambda}\Gamma^\alpha{}_{\gamma\sigma} - \Gamma^\gamma{}_{\beta\sigma}\Gamma^\alpha{}_{\lambda\gamma}. \tag{2.36}$$

$R^\mu{}_{\nu\rho\gamma}$ is an essential tool for measuring the intrinsic curvature of the spacetime, as one can not tell whether a manifold is curved or not by just looking at the line element. This mathematical object is independent of the coordinates and shows the intrinsic curvature of the manifold. As it was discussed earlier, the line element of the spacetime is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In flat spacetime the curvature tensor becomes:

$$R^\alpha{}_{\beta\sigma\lambda} = 0. \tag{2.37}$$

It can be seen from the equation (2.29) that in flat spacetime, the Christoffel symbol and its derivatives vanish. Hence, the condition (2.37) ensures that the spacetime is flat, and vise versa.

Curvature tensor can also be found by another point of view. In flat spacetime, the covariant derivative is simply the usual derivative. It means that the covariant derivatives commute with each other and the order of applying them on a mathematical object does not matter. This is not true in curved spacetime, i.e., $[\nabla_\mu, \nabla_\nu] \neq 0$. Considering v_γ as a four vector and ∇_μ and ∇_ν as the covariant derivatives, the following relation can be found [19]:

$$\nabla_\mu \nabla_\nu v_\gamma - \nabla_\nu \nabla_\mu v_\gamma = R^\rho{}_{\gamma\nu\mu} v_\rho, \tag{2.38}$$

where the curvature tensor can be defined by this equation (2.38). Hence, as in a flat spacetime the covariant derivatives commute, $R^\rho{}_{\gamma\nu\mu}$ becomes zero. Moreover, one can find the correspondence between the parallel transport and the curvature tensor. According to the equation (2.31), the parallel-transportation depends on the path in curved manifolds. As an example, consider the parallel transportation of a four-vector v_μ around a closed loop. The relation between the curvature tensor and the parallel-transported vector at an arbitrary point on the loop is (more detail can be found in [19]):

$$\Delta v^\mu = \frac{-1}{2} R^\mu{}_{\nu\gamma\sigma} v^\nu \oint x^\gamma dx^\sigma, \tag{2.39}$$

where the integral is over a closed loop.

The curvature tensor has important symmetric properties:

$$R_{\alpha\beta\sigma\lambda} = -R_{\beta\alpha\sigma\lambda}, \tag{2.40}$$

$$R_{\alpha\beta\sigma\lambda} = -R_{\alpha\beta\lambda\sigma}, \tag{2.41}$$

$$R_{\alpha\beta\sigma\lambda} = R_{\sigma\lambda\alpha\beta}. \tag{2.42}$$

Beside these properties for the curvature tensor, there is an identity called the *Bianchi identity* that states:

$$\nabla_\gamma R_{\alpha\beta\sigma\lambda} + \nabla_\sigma R_{\alpha\beta\lambda\gamma} + \nabla_\lambda R_{\alpha\beta\gamma\sigma} = 0, \tag{2.43}$$

where ∇ denotes the covariant derivative in curved geometry. Due to these symmetries (2.40)-(2.43), the curvature tensor has only 20 independent components in four-dimensional spacetime.

An important tensor can be calculated from the curvature tensor, which is essential in forming a relationship between the curvature of the spacetime and the matter field. We find this tensor by contracting the first and last indices of the curvature tensor as below [19]:

$$R^{\sigma}{}_{\alpha\beta\sigma} = R_{\alpha\beta}. \tag{2.44}$$

The second rank symmetric tensor $R_{\alpha\beta}$ is called the *Ricci tensor*. Moreover, by applying a contraction on this tensor, we find the *curvature scalar* (*Ricci scalar*) as:

$$R^{\alpha}{}_{\alpha} = R. \tag{2.45}$$

The Kretschmann scalar is calculated from the curvature tensor as $\kappa = R^{\alpha\beta\sigma\lambda}R_{\alpha\beta\sigma\lambda}$. The Kretschmann scalar plays an important role in finding the singularities of the spacetime.

As an example, we rewrite the electrodynamics in curved spacetime. This is an important task as this knowledge is used in the coupling of gravity and electrodynamics. The Lorentz force on a moving electric charge is $\vec{f} = e\vec{E} + \vec{v} \times \vec{B}$, where e is the electron charge, $\vec{v}$ is the velocity of the electron and $\vec{E}$ and $\vec{B}$ are the electric and magnetic field, respectively. This equation is not Lorentz invariant. By introducing an anti symmetric second rank tensor $F_{\mu\nu}$, this equation can be written as [25]:

$$f_\mu = qF_{\mu\nu}u^\nu, \tag{2.46}$$

where f_μ is the electromagnetic force, q is the charge of the particle and $F_{\mu\nu}$ is the electromagnetic field tensor. In an arbitrary frame, this tensor has the following form [19]:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{bmatrix}. \tag{2.47}$$

For finding the electromagnetic field equations, consider a four-current j^μ with components $j^\mu = [c\rho, \vec{j}]$, where ρ is the charge density, c is the speed of light and $\vec{j}$ is the relativistic current in three spatial dimensions. The relation between the four-current and the electromagnetic field tensor is [26]:

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu, \tag{2.48}$$

where μ_0 is the vacuum permeability constant. From the anti-symmetric property of $F_{\mu\nu}$, one finds that $\nabla_\nu \nabla_\mu F^{\mu\nu} = 0$. This condition leads to $\nabla_\mu j^\mu = 0$, which implies the conservation of charge:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0. \tag{2.49}$$

Moreover, the electromagnetic field tensor is related to the electromagnetic gauge field as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.50}$$

From this equation (2.50), an important result can be found:

$$\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0. \tag{2.51}$$

Equations (2.48) and (2.51) lead to the Maxwell's equations. The electromagnetic gauge field in equation (4.15) is a four vector and its components are the electrostatic potential ϕ and the vector potential (in three spatial dimension) $\vec{A}$ as $A^\mu = (\phi/c, \vec{A})$. The observable quantities in the electrodynamics theory are the electric and magnetic fields, which have the following relations with the electromagnetic potential and the vector field:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \tag{2.52}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{2.53}$$

The electromagnetic tensor field can be transferred to any arbitrary Lorentz frame as a second rank tensor:

$$F'_{\mu\nu} = \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu F_{\sigma\rho}. \tag{2.54}$$

2.5 Einstein's Gravitational Field Equations

Having these important mathematical tools that we described earlier, we can find the Einstein tensor. From the Bianchi identity, we find the following equation:

$$\nabla_\alpha (R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R) = 0, \tag{2.55}$$

where R^{ab} is the curvature tensor, R is the Ricci scalar, g^{ab} is the metric tensor and ∇_λ denotes the covariant derivative. We call the expression in the parentheses of the equation (2.55) $G^{\alpha\beta}$ which is the *Einstein tensor*:

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R. \tag{2.56}$$

As the matter indicates the curvature of the spacetime and spacetime tells the matter how to move, we need to find an equation that relates the energy-mass of a matter to the curvature of the spacetime. We constructed the Einstein tensor and now we need to find a proper mathematical object, which contains the energy-mass and relates it to the Einstein tensor. We will see that this equation actually relates the matter to the curvature of the spacetime. The second rank symmetric tensor $T^{\mu\nu}$ is defined as an object which contains all the energies existing at an arbitrary point (except gravity) of the spacetime. This tensor is called the *energy-momentum tensor* and its components are as below [19]:

T^{00} is the energy density;

T^{i0} is the ith component of the momentum density;

T^{0k} is the kth component of the energy flux;

T^{ik} is the ik component of the stress.

A useful example is the energy-momentum tensor for a perfect fluid, which is as follow:

$$T^{\lambda\nu} = (\rho + \frac{p}{c^2})u^\lambda u^\nu - pg^{\lambda\nu}, \tag{2.57}$$

where ρ is the rest frame density, p is the pressure and u indicates the four-velocity of the particle. This equation is useful as it has a lot of applications in cosmology. By letting the pressure $p \to 0$, the equation (2.57) gives the energy-momentum tensor for dust. By dust, we mean particles that have no interaction with each other.

Moreover, as an example, the energy-momentum tensor for the electromagnetic field is:

$$T^{\mu\nu} = \frac{-1}{\mu_0}(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\lambda\gamma}F^{\lambda\gamma}), \tag{2.58}$$

where μ_0 is the vacuum permeability. It can be shown that in curved spacetime, the energy-momentum is conserved. In other words, $\nabla_\mu T^{\mu\nu} = 0$.

By having information about the curvature of the spacetime and the energy-momentum of a matter, we can construct Einstein's gravitational field equations, which relate the curvature to the energy-momentum of the matter. This equation is as below:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}. \tag{2.59}$$

In this equation (2.59), $\kappa = \frac{8\pi G}{c^4}$, G is the gravitational constant and c is the speed of light. By multiplying this equation by $g^{\mu\sigma}$ we find:

$$R^{\sigma}{}_{\nu} - \frac{1}{2}\delta^{\sigma}_{\nu}R = -\kappa T^{\sigma}{}_{\nu}. \tag{2.60}$$

A useful version of the Einstein gravitational field equation is found by contracting σ and ν in equation (2.60):

$$R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \tag{2.61}$$

It can be seen from equation (2.59) that Einstein's gravitational field equations are non-linear and there are 10 independent components for these equations. The weak-field limit of the Einstein equations leads to the Poisson's equation in Newtonian gravity:

$$\nabla^2\Phi = 4\pi G\rho, \tag{2.62}$$

where Φ is the gravitational potential in classical mechanics.

It is useful to study the Einstein equation in empty spacetime. The energy-momentum tensor $T_{\mu\nu}$ includes all forms of energy and momentum. Hence, in an empty spacetime, this tensor is $T_{\mu\nu} = 0$. Applying this condition to equation (2.59), one finds that the Ricci tensor is zero in empty spacetime $R_{\mu\nu} = 0$. In other words, in that region, the spacetime is free of matter and other energies (such as radiation). We find an important result by comparing the Ricci tensor and the curvature tensor in empty spacetime. In two or three dimensional spacetime, the condition of $R_{\mu\nu} = 0$ ensures that the curvature tensor must vanish as well. But in four or more dimensional spacetime, the curvature tensor has 20 independent components, while there are only 10 independent field equations. Hence, we can have a non-vanishing curvature tensor in four or more dimensional spacetime even if the Ricci tensor is zero (empty spacetime). This means that gravitational fields can exist even in empty spacetime [19].

The Einstein gravitational field equations are not unique. As $\nabla_\nu G_{\mu\nu} = 0$, other terms can be added to the Einstein tensor, which leaves this condition intact. In the time when Einstein proposed his theory of relativity, it was believed that the universe should obey a static model. Einstein's equations do not imply such a property for the universe. Therefore, Einstein added a term to his equation to have a static model of the universe. This new term is $\Lambda g_{\mu\nu}$, where Λ is called the *cosmological constant*. As $\nabla_\mu g^{\mu\nu} = 0$, the new added term does not alter the $\nabla_\nu G_{\mu\nu} = 0$ condition. Therefore, the field equations become:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{2.63}$$

When Hubble discovered the expansion of the universe, Einstein said that his introduction of the cosmological constant was his biggest blunder. Nowadays, we have a different point of view about the cosmological constant. The cosmological constant is considered as a universal constant and has the interpretation of the energy density of the vacuum. Consider ρ as the energy density of the vacuum. The cosmological constant can be written in terms of ρ as:

$$\frac{\Lambda c^4}{8\pi G} = \rho^{(vac)}c^2. \tag{2.64}$$

The energy-momentum tensor for the vacuum is defined in terms of the cosmological constant as:

$$T^{(vac)}_{\mu\nu} = \rho^{(vac)}c^2 g_{\mu\nu} = \frac{\Lambda c^4}{8\pi G}g_{\mu\nu}. \tag{2.65}$$

Hence, we write the Einstein field equation as [19]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa(T_{\mu\nu} + T^{vac}_{\mu\nu}), \tag{2.66}$$

where $T^{vac}_{\mu\nu}$ is given in equation (2.65) .

The approach to the Einstein's gravitational field equation that we showed is not unique and other approaches to gravity exist. One of the different approaches is the Brans-Dicke theory [27]. In this theory, gravity is again described with respect to the curvature of the spacetime and also the equivalence principle is considered similar to the general relativity. However, a scalar field is defined Φ which can be fixed with respect to the energy-momentum tensor and a coupling constant λ. The gravitational constant G is being fixed by the scalar field Φ. Moreover, the field equations for gravity is found from the relation between the curvature of the spacetime and the energy-momentum of the matter ($T^{M}_{\mu\nu}$), plus the energy-momentum of the scalar field ($T^{\phi}_{\mu\nu}$). In other words, the scalar field implies the coupling strength of the matter to gravity [19, 28]. The equations of motion are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{-8\pi}{c^4 \phi} (T^{M}_{\mu\nu} + T^{\phi}_{\mu\nu}), \tag{2.67}$$

$$\partial_\mu \partial^\mu \phi = -4\pi\lambda T^{(M)\mu}{}_{\mu}. \tag{2.68}$$

It is worth noting that the gravitational constant in the Brans-Dicke theory is time dependent.

Another example of the different approaches to gravity would be the Torsion theories, in which the manifold of the spacetime is not torsion-less anymore. This implies that the Christoffel symbol is not symmetric anymore, as the torsion tensor is defined as:

$$T^{\mu}{}_{\nu\lambda} = \Gamma^{\mu}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\lambda\nu}. \tag{2.69}$$

Now that a relationship between the curvature of the spacetime and the existed energy-momentum in a region of spacetime is found by Einstein's gravitational field equations, the challenge of solving them is ahead. It can be seen from the equation (2.5) that these equations are highly non-linear. In other words, solving the equations in a general way with an analytical approach can not be done. Therefore, it is easier to find the exact solutions to the Einstein equations under some constraints and symmetries. One of the first exact solutions to these equations was done by Schwarzschild.

2.6 Schwarzschild Geometry

One of the first exact solutions to the Einstein field equations was proposed by Schwarzschild in 1916. He found the metric in the static empty spacetime around a spherically symmetric matter distribution. Static metric means that the components of the metric tensor are independent of time and there is no rotation. By considering these assumptions, we can find an analytical solution to the Einstein gravitational field.

The general form of the line element is $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. By applying the demand of the static spherically symmetric, the line element is written as [29]:

$$ds^2 = f(r)dt^2 - h(r)dr^2 - r^2 d\Omega^2, \tag{2.70}$$

where $f(r)$ and $h(r)$ are functions of r, and $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$. As the Schwarzschild's metric is in empty spacetime, one can use the condition of $R_{\mu\nu} = 0$, which ensures the emptiness of the spacetime. By considering this condition, we write the line element (2.70) as:

$$ds^2 = c^2(1 - \frac{2\mu}{r})dt^2 - (1 - \frac{2\mu}{r})^{-1}dr^2 - r^2d\theta^2 - r^2 \sin^2 \theta d\phi^2. \tag{2.71}$$

This line element (2.71) is called the *Schwarzschild metric* in empty spacetime around a spherical body of mass M. In this equation (2.71), $\mu = \frac{GM}{c^2}$ and G is the gravitational constant, M is a spherically symmetric mass and c is the speed of light. This metric is valid down to the surface of the massive body, as one condition that is used to construct the Schwarzschild metric is the empty spacetime condition.

As it can be seen from equation (2.71), the Schwarzschild metric poses a singularity at $r = 2\mu$. This is called the Schwarzschild radius. It is worth noting that this singularity is a coordinate singularity, as one can propose a different coordinate for this metric to remove the singularity at $r = 2\mu$. Moreover, by calculating the Ricci scalar, we find that this radius is not an intrinsic singularity and it can be removed. The only intrinsic singularity of the Schwarzschild metric is located at $r = 0$

By having the Schwarzschild metric, the geodesic of particles can be found. The concept of the Killing vector is used in differential geometry for indicating the symmetries. We introduce the Killing vector ζ which obeys the Killing equation:

$$\nabla_\nu \zeta_\mu + \nabla_\mu \zeta_\nu = 0, \tag{2.72}$$

where ∇ is the covariant derivation and $\nabla_\nu \zeta_\mu$ is an antisymmetric entity. There is an important theorem, which states that in any geometry, the product of the Killing vector ζ and the four-momentum p (which is a tangent vector to the geodesic) leads to a constant entity for the geodesic motion [22]. Moreover, the line element of the Schwarzschild geometry in equation (2.71) implies that the translation $t' = t + \delta t$ and $\phi' = \phi + \delta\phi$ does not affect the geometry. Therefore, the conjugate momentum p_t and p_ϕ are conserved and lead to the constants of the motion. These conjugate momentums are:

$$p_t \equiv -E, \tag{2.73}$$

and

$$p_\phi \equiv \pm L, \tag{2.74}$$

where E and L are energy and angular momentum, respectively.

We find the geodesics in the Schwarzschild geometry by considering the Lagrangian $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, where:

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}, \tag{2.75}$$

and λ is an affine parameter. According to the Schwarzschild metric (2.71), we construct the Lagrangian from equation (2.33):

$$L = c^2(1 - \frac{2\mu}{r})(\frac{dt}{d\lambda})^2 - \frac{1}{1 - \frac{2\mu}{r}}(\frac{dr}{d\lambda})^2 - r^2(\frac{d\theta}{d\lambda})^2 - r^2 \sin^2 \theta(\frac{d\phi}{d\lambda})^2. \tag{2.76}$$

From the Euler-Lagrange equation, we find the following geodesic equations [19]:

$$(1 - \frac{2\mu}{r})\frac{dt}{d\lambda} = A, \tag{2.77}$$

$$r^2 \sin^2\theta \frac{d\phi}{d\lambda} = B, \tag{2.78}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}(\frac{dr}{d\lambda})(\frac{d\theta}{d\lambda}) - \sin\theta\cos\theta(\frac{d\phi}{d\lambda})^2 = 0, \tag{2.79}$$

$$\frac{1}{1-\frac{2\mu}{r}}\frac{d^2r}{d\lambda^2} - \frac{1}{(1-\frac{2\mu}{r})^2}\frac{\mu}{r^2}(\frac{dr}{d\lambda})^2 + \frac{\mu c^2}{r^2}(\frac{dt}{d\lambda})^2 - r^2(\frac{d\theta}{d\lambda})^2 - r^2\sin^2\theta(\frac{d\phi}{d\lambda})^2 = 0, \tag{2.80}$$

where A and B are constants.

According to the information given above, we find the trajectories for massive particles and photons. We separate the trajectories to radial motion and circular motion for both massive particles and photons. For a massive particle, the radial motion, where ϕ is a constant, is as below [19]:

$$(\frac{dr}{d\tau})^2 = c^2(A^2-1) + \frac{2Gm}{r}, \tag{2.81}$$

where τ is the proper time of the particle and A is a constant of the motion that corresponds to the energy of the particle. A very interesting feature can be seen by considering a special case, where a particle is being dropped from infinity, the equation of motions become:

$$\frac{dt}{d\tau} = \frac{1}{1-2\mu/r}, \tag{2.82}$$

$$\frac{dr}{d\tau} = -\sqrt{2\mu c^2/r}. \tag{2.83}$$

From these two equations (2.82) and (2.83), an explicit formula is found for t and τ (due to the length of these calculation, we do not mention them here. More details in [22]). By letting the radial coordinate to $r \to 2\mu$ (Schwarzschild radius), the time of the particle measured by a stationary observer at a large distance goes to infinity $t \to \infty$, while the proper time of the particle becomes a finite number. It means that particle can actually pass the Schwarzschild radius.

The circular motion of a massive particle is [19]:

$$(\frac{d\phi}{d\tau})^2 = \frac{\mu c^2}{(r-3\mu)r^2}. \tag{2.84}$$

An important result is that the radius of the innermost stable circular orbit for a massive particle is at $r = 6\mu$.

For photons or any other mass-less particles, the radial and orbital motion can be found from the geodesic equations. We only mention the results and do not mention the process of deriving these equations. We find the radial and circular motion for a photon as below, respectively [21]:

$$\frac{dr}{dt} = \pm c(1-\frac{2\mu}{r}), \tag{2.85}$$

$$(\frac{du}{d\phi})^2 + u^2 = \frac{1}{b^2} + 2\mu u^3, \tag{2.86}$$

where $u = 1/r$ and b is a constant that is called the impact parameter.

It is worth to mention that in the Schwarzschild metric if a massive body gets compact enough under a few circumstances in a way that the radius of its matter gets smaller than the Schwarzschild radius of the body, it becomes a very interesting object, which is called the Schwarzschild Black Hole. We do not study the Schwarzschild black holes and their properties here.

There are other exact solutions to the general relativity and we mention two of them briefly.

Since we derived the Schwarzschild metric by considering a static geometry, the Schwarzschild's geometry fails to explain the geometry of a rotating massive body. The metric that explains a rotating body should also depends on the angular momentum. *Kerr* geometry explains the geometry of a rotating body in an empty spacetime. The metric of such a metric is as below [19, 30]:

$$ds^2 = \frac{\rho^2 \Delta}{\Sigma^2} c^2 dt^2 - \frac{\Sigma^2 \sin^2\theta}{\rho^2} (d\phi - \omega dt)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2. \tag{2.87}$$

In equation (2.87), $\mu = GM/c^2$ and Σ, ω, ρ and Δ are given as:

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta, \tag{2.88}$$

$$\omega = \frac{2\mu c r a}{\Sigma^2}, \tag{2.89}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta, \tag{2.90}$$

$$\Delta = r^2 - 2\mu r + a^2. \tag{2.91}$$

Geodesics can be found analytically from the Kerr metric. Moreover, the Kerr black hole leads to important results [31], which we do not study them here.

Another exact solution to the Einstein gravity is the Reissner–Nordstrom metric [32], which indicates the gravitational field around a non-rotating charged spherically symmetric massive body [33]. The line element of this geometry is [34]:

$$ds^2 = -c^2\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)dt^2 + \frac{1}{1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}} dr^2 + r^2 d\Omega^2, \tag{2.92}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and r_s and r_Q are given as below:

$$r_s = 2GM/c^2, \tag{2.93}$$

$$r_Q^2 = \frac{GQ^2}{kc^4}. \tag{2.94}$$

In the last equation (2.94), $k = \frac{1}{4\pi\epsilon_0}$ is the Coulomb constant. Moreover, the electromagnetic gauge field is $A_\mu = (Q/r, 0, 0, 0)$.

There are experimental evidence that confirm the theory of general relativity. The precession of the planetary orbits such as Mercury is one example that experiments confirm the predicted results from general relativity. Another important result is the bending of light by a massive object. The shape equation for the trajectory of a photon in Schwarzschild geometry is:

$$\frac{d^2u}{d\phi^2} + u = 3GMu^2/c^2, \tag{2.95}$$

Figure 2.1: First image of a super massive black hole which was taken by Event Horizon Telescope. This observation is of the center of the galaxy M87 [20].

where $u \equiv \frac{1}{r}$. Hence, the total deflection of light is found to be [19]:

$$\Delta\phi = \frac{4GM}{c^2 b}, \tag{2.96}$$

where b is the impact parameter. This phenomena is experimentally confirmed [35].

A recent experimental observation that confirmed the predictions of general relativity is the capturing of the image of a supermassive black hole at the core of the supergiant elliptical galaxy Messier 87 [36], which was released by the Event Horizon Telescope in 2019 (Figure 2.1).

General relativity plays an important role in modeling the behaviour of the universe. One of the most interesting models in cosmology is the Friedmann-Robertson-Walker (FRW) geometry, which is in agreement with many experiments. We study this model briefly in the next section.

2.7 Friedmann-Robertson-Walker Geometry

As we look at the universe at the larger scales, we realize that the distribution of matter looks uniform. Moreover, some evidence such as the constancy of the temperature of the cosmic microwave background in different directions of the sky, suggest that the universe is isotropic. Hence, by adopting the large scale point of view, we assume the cosmological principle that suggests that the universe looks the same from all positions at any particular time. Moreover, all directions in space are equivalent at any point [19]. A particular time in this method is defined as a spacelike hypersurface. Also the idealized concept of the fundamental observer is assumed, which in his frame measures no dipole moment of the cosmic microwave background radiation, i.e., the observer has no motion relative to the overall cosmological fluid associated with the smeared-out motion of all the galaxies [19].

By these assumptions, the Friedmann-Robertson-Walker (FRW) metric is found as [22]:

$$ds^2 = c^2 dt^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2], \tag{2.97}$$

where $R(t)$ is the scale function, $k \in \{-1, 0, 1\}$ which depends on whether the spatial part of the spacetime has negative, zero or positive curvature.

We write the FRW metric in a more convenient way as:

$$ds^2 = c^2dt^2 - R^2(t)[d\chi^2 + S^2(\chi)d\Omega^2], \tag{2.98}$$

where χ is the new radial coordinate defined by $r = \sin\chi$ and the function $S(\chi)$ is given as below, depending on k [19]:

$$k = 1 : S(\chi) = \sin\chi, \tag{2.99}$$

$$k = 0 : S(\chi) = \chi, \tag{2.100}$$

$$k = -1 : S(\chi) = \sinh\chi. \tag{2.101}$$

From this discussion one finds the constraints of homogeneity and isotropy very important, as these assumptions are partially in agreement with the observed phenomena. In the next section, we classify the homogeneous spaces by following Bianchi's method and find important geometries from this classification.

3 Bianchi Geometry

In this chapter, we classify the homogeneous spaces by following Bianchi's procedure. Our research is done based on the Bianchi type IX geometry, which is included in this classification of homogeneous spaces. Hence, we study the Bianchi type IX geometry in more details and discuss the properties of its well-known subspaces such as Eguchi-Hanson type I and type II, Atiyah-Hitchin space and Taub-NUT geometry.

3.1 Classification of the Bianchi Spaces

As we discussed in the previous chapter, the symmetries that one considers for the metric make is possible to find the exact solutions to the highly non-linear Einstein's equations. One of the most important symmetries is to consider the metric to be homogeneous and isotropic. These assumptions are vital in cosmological models such as the Friedmann-Robertson-Walker model (FRW), which is in agreement with the observed phenomenology such as the constancy of the temperature of the Cosmic Background Radiation (CBR) [37, 38]. Hence, classifying the homogeneous spaces and studying the dynamical behaviour of them is essential for studying the structure of the universe. The classification of the homogeneous spaces was done in 1897 by Bianchi and later on used in cosmology by Lifschitz, Belinski and Khalatnikov [39].

By the assumptions of homogeneity and isotropy of space, the metric of the spacetime can be determined. It is worth noting that homogeneity does not necessarily imply isotropy, while isotropy about every point ensures the homogeneity [19]. By relaxing the isotropic restriction, we find a classification of homogeneous spaces in this section.

A space is homogeneous if different points of the space are identical. Homogeneity of space leads to the identical metric properties at all points of the manifold. In other words, a homogeneous space admits a set of transformations that enable one to bring any given point to the position of any other point. In group theory terminology, each point of the spacetime is equivalent under the action of the group in a homogeneous space [40]. We start with a group of transformations and investigate the Lie algebra of this group:

$$x^\mu \to x'^\mu = T^\mu(x, \zeta), \tag{3.1}$$

where the set of $\{\zeta^a | a \in 1, ..., r\}$ are r independent variables that parameterize the group. Consider an infinitesimal transformation:

$$x^\mu \to x'^\mu = T^\mu(x, \zeta_0 + \delta\zeta) \approx x^\mu + \xi^\mu_a(x)\delta\zeta^a = (1 + \delta\zeta^a \xi_a)x^\mu, \tag{3.2}$$

where ζ_0 corresponds to the identical transformation, $T^\mu(x,\zeta_0) = x^\mu$. Moreover, in equation (3.2) we considered [41]:

$$(\frac{\partial T^\mu}{\partial \zeta^a})(x,\zeta_0) \equiv \xi^\mu_a(x). \tag{3.3}$$

We defined the r first order differential operators $\{\xi_a\}$ in equation (3.2) in correspondence with the r vectorial fields with components $\{\xi^\mu_a\}$ with the relation $\xi_a = \xi^\mu_a \frac{\partial}{\partial x^\mu}$, which are the killing generating vectors. The Lie algebra that applies to the killing vectors in the commutation relation form is:

$$[\xi_a, \xi_b] = C^c_{ab}\xi_c. \tag{3.4}$$

In equation (3.4), C^c_{ab} is the structure constant and by extending this formalism, we introduce the basis $\{e_\alpha\}$ for the Lie algebra with the commutation relation as:

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma, \tag{3.5}$$

and define the symmetric quantity as $\gamma_{\alpha\beta} = \gamma_{\beta\alpha} = C^\gamma_{\alpha\sigma} C^\sigma_{\beta\gamma}$. The relation in equation (3.5) defines a group of transformations (non-Abelian) which represents the spatially homogeneous part of the spacetime. The metric tensor can be defined with respect to the bases $\{e_\alpha\}$ as:

$$g_{\alpha\beta} = e_\alpha e_\beta, \tag{3.6}$$

where $\{\alpha, \beta \in \{0,1,2,3\}\}$. Each class of the equivalence Lie group needs to be indicated by only one representative group. Bianchi classification indicates all symmetries for the homogeneous spaces, which in tridimensional (spatial section of the spacetime) is in analogy to the Friedmann-Robertson-Walker model of the curvature with $k = \pm 1, 0$ that indicates the distention of homogeneous and isotropic spaces [41].

A well-known approach in cosmology is to consider a synchronized system with a unique time t and label the hypersurfaces with this synchronized time coordinate t. Each hypersurface is orthogonal to the observer's worldline, i.e., the metric tensor has the following property [19]:

$$g_{i0} = 0, \tag{3.7}$$

where $i \in \{1,2,3\}$.

Moreover, we introduce the comoving coordinates (x^1, x^2, x^2) for the fundamental observer to be fixed along the worldline. Hence, the line element can be written as:

$$ds^2 = c^2 dt^2 - dl^2, \tag{3.8}$$

where the spatial metric is:

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta. \tag{3.9}$$

In equation (3.9), $\gamma_{\alpha\beta}$ is the metric spatial sector. Since the comoving coordinates are constant along the worldline, the line element (3.8) can be rewritten as:

$$ds = c d\tau = c dt, \tag{3.10}$$

and therefore, the proper time along the worldline is equal to the synchronized time t. It is worth noting that $x^i = constant$ are geodesics of this line element. The four-velocity of a fundamental observer in this comoving coordinate is $u^\mu = (1, 0, 0, 0)$. As any vector lying in the introduced hypersurfaces has the form $\eta^\mu = (0, \eta^1, \eta^2, \eta^3)$, the four-velocity is orthogonal to the hypersurface. The geodesic equation in this system is:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0. \tag{3.11}$$

The components of the Einstein's equations calculated from the line element (3.8) are [41]:

$$R^0{}_0 = \frac{-1}{2}\frac{\partial \kappa^\alpha{}_\alpha}{\partial t} - \frac{1}{4}\kappa^\beta{}_\alpha \kappa^\alpha{}_\beta = 8\pi G\frac{1}{2}(2T^0{}_0 - T), \tag{3.12}$$

$$R^0{}_\alpha = \frac{1}{2}(\kappa^\beta{}_{\alpha;\beta} - \kappa^\beta{}_{\beta;\alpha}) = 8\pi G T^0{}_\alpha, \tag{3.13}$$

$$R^\beta{}_\alpha = -P^\beta{}_\alpha - \frac{1}{(4\gamma)^{1/2}}\frac{\partial}{\partial t}(\sqrt{\gamma}\kappa^\beta{}_\alpha) = 8\pi G\frac{1}{2}(2T^\beta{}_\alpha - \delta^\alpha_\beta T), \tag{3.14}$$

where G is the gravitational constant, $T = det[T_{\alpha\beta}]$, $\kappa_{\alpha\beta} = \frac{\partial \gamma_{\alpha_\beta}}{\partial t}$ and $\gamma = det[\gamma_{\alpha\beta}]$. In equations (3.12)-(3.13), $T_{\alpha\beta}$ and $P_{\alpha\beta}$ are energy-momentum tensor and tridimensional Ricci tensor from the metric $\gamma_{\alpha\beta}$, respectively.

By considering a tetradic basis on each point of the spacetime, we can rewrite the physical quantities in a simpler way. We show the tetradic basis of the four linearly independent vectors on each point as $e^i_{(a)}$, where $a \in \{1, ..., 4\}$ indicates the tetradic and i represents the tensorial part. These bases satisfy the orthogonality condition:

$$e^{(a)}_i e^k_{(a)} = \delta^k_i. \tag{3.15}$$

Moreover, vectors and tensor can be expanded in terms of the tetradic bases [42]:

$$V_{(a)} = e_{(a)j}V^j = e^j_{(a)}V_j, \tag{3.16}$$

$$T_{(a)(b)} = e^i_{(a)}e^j_{(b)}T_{ij}, \tag{3.17}$$

where $V_{(a)}$ and $T_{(a)(b)}$ represent a vector and a tensor in tetradic bases, respectively. The tetradic indices can be lowered and raised by $\eta_{(ab)}$, which is defined by $e^i_{(a)}e_{(b)i} = \eta_{ab}$. For example:

$$e_{(b)i} = \eta_{(a)(b)}e^{(a)}_i, \tag{3.18}$$

$$e^{(b)}_i = \eta^{(a)(b)}e_{(a)i}. \tag{3.19}$$

We consider the signature of $\eta_{(ab)}$ to be (+,-,-,-). According to the equations (3.18) and (3.19), we calculate $V^{(a)}$ as:

$$V^{(a)} = \eta^{(a)(b)}V_{(b)}. \tag{3.20}$$

Moreover, the metric tensor in terms of the tetradic bases is $g_{ij} = e_{(a)i}e^{(a)}_j = \eta_{(ab)}e^{(a)}_i e^{(b)}_j$. Hence, the line element becomes [41]:

$$ds^2 = \eta_{ab}(e^{(a)}_i dx^i)(e^{(b)}_k dx^k). \tag{3.21}$$

It is worth noting that $dx^{(a)} = e_i^{(a)} dx^i$ are not exact differentials of functions of the coordinates in general. Considering the tetradic bases $e_{(a)}$ of tangent vectors, the covariant derivative is expressed as below:

$$e_{(a)} = e^i_{(a)} \frac{\partial}{\partial x^i}, \tag{3.22}$$

so the derivative of a vector is written as:

$$V_{(a),(b)} = e^j_{(a)} V_{j;i} e^i_{(b)} + e_{(a)k;i} e^i_{(b)} e^k_{(c)} V^{(c)}, \tag{3.23}$$

where we used equations (3.22) and (3.16). The second term in the parenthesis in equation (3.23) is called the *Ricci's rotation coefficients* and is denoted by γ_{abc}, which is antisymmetric with respect to its first two indices:

$$\gamma_{abc} = -\gamma_{bac}. \tag{3.24}$$

The linear combination of the Ricci's rotation coefficient is defined as $\lambda_{abc} = \gamma_{abc} - \gamma_{acb}$, which is antisymmetric with respect to its last two indices $\lambda_{abc} = -\lambda_{acb}$. The next step is to find the structure constants in a way that the metric becomes invariant under the homogeneity constraint. The Lie algebra in the tetradic bases is:

$$[e_{(a)}, e_{(b)}] = C^{(c)}_{(a)(b)} e_{(c)}, \tag{3.25}$$

where the structure constants of the group of transformation is denoted by $C^{(c)}_{(a)(b)}$ and has the antisymmetric properties $C^{(c)}_{(a)(b)} = -C^{(c)}_{(b)(a)}$. Different quantities can be rewritten in terms of the structure constants and Ricci rotation coefficients. We find a classification of the homogeneous spaces with these tools, similar to the Bianchi's approach.

In order to have an invariant line element dl (equation (3.9)) under the transformation of its group of motion, $\gamma_{\alpha\beta}$ must be the same under the homogeneity constraint. We write the spatial part of the line element as:

$$dl^2 = \eta_{ab} (e^{(a)}_\mu dx^\mu)(e^{(b)}_\nu dx^\nu), \tag{3.26}$$

which makes $\gamma_{\mu\nu}$ to be $\gamma_{\mu\nu} = \eta_{ab} e^{(a)}_\mu e^{(b)}_\nu$. The relations between $e^\mu_{(a)}$ and $e^{(a)}_\mu$ in a very specific case are [42]:

$$e_{(1)} = \frac{1}{\nu} e^{(2)} \times e^{(3)}, \tag{3.27}$$

$$e_{(2)} = \frac{1}{\nu} e^{(3)} \times e^{(1)}, \tag{3.28}$$

$$e_{(3)} = \frac{1}{\nu} e^{(1)} \times e^{(2)}, \tag{3.29}$$

where $\nu = e^{(1)} . e^{(2)} \times e^{(3)}$ and $e^{(a)}$ and $e_{(a)}$ are interpreted as Cartesian vectors with components $e^{(a)}_\mu$ and $e^\mu_{(a)}$, respectively.

Equation (3.26) also implies that $e^{(a)}_\mu dx^\mu$ is invariant under such transformations. Hence [42]:

$$e^{(a)}_\mu (x) dx^\mu = e^{(a)}_\mu (x') dx'^\mu. \tag{3.30}$$

A system of differential equations for determining $x'^{\nu}(x)$ can be obtained from (3.30) as:

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = e^{\nu}_{(a)}(x')e^{(a)}_{\mu}(x). \tag{3.31}$$

The equation (3.31) is integrable if:

$$\frac{\partial^2 x'^{\nu}}{\partial x^{\mu}\partial x^{\gamma}} = \frac{\partial^2 x'^{\nu}}{\partial x^{\gamma}\partial x^{\mu}}, \tag{3.32}$$

which is called the Schwartz's condition [43]. Substituting (3.31) in (3.32) and using some algebra, it becomes:

$$[\frac{\partial e^{\nu}_{(a)}(x')}{\partial x'^{\sigma}} e^{\sigma}_{(b)}(x') - \frac{\partial e^{\nu}_{(b)}(x')}{\partial x'^{\sigma}} e^{\sigma}_{(a)}(x')] e^{(b)}_{\gamma}(x) e^{(a)}_{\mu}(x) = e^{\nu}_{(a)}(x')[\frac{\partial e^{(a)}_{\gamma}(x)}{\partial x^{\mu}} - \frac{\partial e^{(a)}_{\mu}(x)}{\partial x^{\gamma}}]. \tag{3.33}$$

After using the properties of the tetradic base and some algebra, we force both sides of the equation (3.33) to be equal to each other and equal to the same constant:

$$(\frac{\partial e^{(c)}_{\mu}}{\partial x^{\nu}} - \frac{\partial e^{(c)}_{\nu}}{\partial x^{\mu}}) e^{\mu}_{(a)} e^{\nu}_{(b)} = C^{c}_{ab}, \tag{3.34}$$

where C^{c}_{ab} is the structure constant. The uniformity condition is obtained by multiplying (3.34) by $e^{\gamma}_{(c)}$ [41]:

$$e^{\mu}_{(a)} \frac{\partial e^{\gamma}_{(b)}}{\partial x^{\mu}} - e^{\nu}_{(a)} \frac{\partial e^{\gamma}_{(a)}}{\partial x^{\nu}} = C^{c}_{ab} e^{\gamma}_{(c)}. \tag{3.35}$$

By defining a linear operator as $X_a = e^{\mu}_{(a)} \frac{\partial}{\partial x^{\mu}}$, we rewrite the equation (3.35):

$$[X_a, X_b] = C^{c}_{ab} X_c, \tag{3.36}$$

where the commutation $[X_a, X_b]$ implies $[X_a, X_b] = X_a X_b - X_b X_a$.

We use the Jacobi identity to express the homogeneity:

$$[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0. \tag{3.37}$$

We can write this equation (3.37) in terms of the structure constants:

$$C^{h}_{ab} C^{d}_{ch} + C^{h}_{bc} C^{d}_{ah} + C^{h}_{ca} C^{d}_{bh} = 0, \tag{3.38}$$

where the two index structure constant is defined as the dual of C^{c}_{ab} as follow:

$$C^{c}_{ab} = \epsilon_{abd} C^{dc}. \tag{3.39}$$

In equation (3.39) ϵ is the Levi-Civita pseudo tensor and equation (3.37) can be written as:

$$\epsilon_{abc} C^{bc} C^{ad} = 0. \tag{3.40}$$

This mathematical tools and their relations enable us to classify the non-equivalent homogeneous spaces by using non-equivalent combinations of the constants C^{ab} [42].

$$[X_1, X_2] = -aX_2 + n_3 X_3, \tag{3.41}$$

$$[X_2, X_3] = n_1 X_1, \tag{3.42}$$

$$[X_3, X_1] = n_2 X_2 + a X_3. \tag{3.43}$$

The constants (n_1, n_2, n_3) and a in equations (3.41)-(3.43) are related to the structure constants. Non-equivalent structure constants that lead to the non-equivalent homogeneous spaces is classified in Bianchi classification as follow [42]:

Type I: $a = 0$ and $(n_1, n_2, n_3) = (0, 0, 0)$,

Type II: $a = 0$ and $(n_1, n_2, n_3) = (1, 0, 0)$,

Type III: $a = 1$ and $(n_1, n_2, n_3) = (0, 1, -1)$,

Type IV: $a = 1$ and $(n_1, n_2, n_3) = (0, 0, 1)$,

Type V: $a = 1$ and $(n_1, n_2, n_3) = (0, 0, 0)$,

Type VI: $a = 0$ and $(n_1, n_2, n_3) = (1, -1, 0)$,

Type VII: $a = 0$ and $(n_1, n_2, n_3) = (1, 1, 0)$,

Type VIII: $a = 0$ and $(n_1, n_2, n_3) = (1, 1, -1)$,

Type IX: $a = 0$ and $(n_1, n_2, n_3) = (1, 1, 1)$.

Each type of the Bianchi spaces has its own properties and applications in different theories. We provide a few examples for each of these spaces:

-Bianchi type I: quantum loop gravity and supersymmetric Bianchi type in cosmology [44, 45],

-Bianchi type II: string cosmological model [46],

-Bianchi type III: cosmological model in $f(R,T)$ theory of gravity and cosmological model in string theory [47, 48],

-Bianchi type IV and V: the study of dark energy based on this space [49, 50],

-Bianchi type VI: cosmological model in $f(R)$ and $f(R,T)$ theory [51, 52],

-Bianchi type VII: exact Branes-Dicke perfect fluid solutions based on this space [53],

-Bianchi type VIII: cosmological models with rotation dark energy [54],

-Bianchi type IX: string cosmological models [55].

Among all of these Bianchi type spaces, we focus on the Bianchi type IX geometry. We find the exact solutions to the Einstein-Maxwell-dilaton theory based on this geometry.

3.2 Bianchi type IX Geometry

The Bianchi type IX geometry has essential properties and has been used in different areas of gravitational physics. This metric can be written with an SO(3) or SU(2) isometry group as [56]:

$$ds^2_{B.IX} = e^{2f(\eta)}\sigma_1^2 + e^{2h(\eta)}\sigma_2^2 + e^{2g(\eta)}\sigma_3^2 + e^{2\{f(\eta)+h(\eta)+g(\eta)\}}d\eta^2. \tag{3.44}$$

In this equation (3.44), σ_i's are a basis of SO(3) one-forms which satisfy $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$ [57]. These one-forms are:

$$\sigma_1 = d\psi + \cos\theta d\phi, \tag{3.45}$$

$$\sigma_2 = \cos\psi \sin\theta d\phi - \sin\psi d\theta, \tag{3.46}$$

$$\sigma_3 = \sin\psi \sin\theta d\phi + \cos\psi d\theta, \tag{3.47}$$

and are called Maurer-Cartan one-forms. The periodic Euler angles θ, ϕ and ψ form an SO(3) group and their periodicities are π, 2π and 4π, respectively.

The Bianchi type IX metric in equation (3.44) satisfies the Einstein equations:

$$\varepsilon_{rr} = 2(\frac{dg}{d\eta})(\frac{df}{d\eta}) + (\frac{dg}{d\eta})^2 - e^{2f+2h} + \frac{1}{4}e^{2h+4f-2g} = 0, \tag{3.48}$$

$$\varepsilon_{\theta\theta} = e^{-2(f+h+g)}(\frac{-1}{4}e^{4f+2h} + (\frac{d^2f}{d\eta^2} + \frac{d^2g}{d\eta^2} + (-\frac{dh}{d\eta} - \frac{dg}{d\zeta})\frac{df}{d\eta} - \frac{dh}{d\eta}\frac{dg}{d\eta})e^{2g}) = 0, \tag{3.49}$$

$$\varepsilon_{\psi\psi} = \frac{e^{2g}(\frac{dg}{d\eta})^2 - 2e^{2g}\frac{dg}{d\eta}\frac{df}{f\eta} - 2e^{2g}\frac{dg}{d\eta}\frac{dh}{d\eta} + 2e^{2g}\frac{d^2g}{d\eta^2} + \frac{3}{4}e^{4f+2h} - e^{2(f+h+g)}}{e^{4g+2h}} = 0. \tag{3.50}$$

The other non-zero components of the Einstein's equations are $\varepsilon_{\phi\phi}$ and $\varepsilon_{\phi\psi}$, which due to their length, we do not mention them. Combining the results, we write them down in a more compact form as:

$$-4(\frac{df}{d\eta}\frac{dh}{d\eta} + \frac{dh}{d\eta}\frac{dg}{d\eta} + \frac{dg}{d\eta}\frac{df}{d\eta}) = (e^{4f} + e^{4h} + e^{4g}) - 2(e^{2(f+h)} + e^{2(f+g)} + e^{2(g+h)}), \tag{3.51}$$

$$-2\frac{d^2f}{d\eta^2} = (e^{2h} - e^{2g})^2 - e^{4f}, \tag{3.52}$$

$$-2\frac{d^2h}{d\eta^2} = (e^{2g} - e^{2f})^2 - e^{4h}, \tag{3.53}$$

$$-2\frac{d^2g}{d\eta^2} = (e^{2f} - e^{2h})^2 - e^{4g}. \tag{3.54}$$

Self-duality of the curvature gives first order differential equations for $f(\eta)$, $h(\eta)$ and $g(\eta)$ [56]:

$$2\frac{df}{d\eta} = e^{2h} + e^{2g} - e^{2f} - 2\lambda_1 e^{h+g}, \tag{3.55}$$

$$2\frac{dh}{d\eta} = e^{2g} + e^{2f} - e^{2h} - 2\lambda_2 e^{f+g}, \tag{3.56}$$

$$2\frac{dg}{d\eta} = e^{2f} + e^{2h} - e^{2g} - 2\lambda_3 e^{f+h}. \tag{3.57}$$

In equations (3.55)-(3.57), the constants $\{\lambda_i | i \in 1, 2, 3\}$ satisfy the relation $\lambda_i\lambda_j = \epsilon_{ijk}\lambda_k$, which leads to five different choices for the set (λ_1,λ_2,λ_3) [58].

By choosing (λ_1,λ_2,λ_3)=$(1, 1, 1)$ and substituting them in equations (3.55)-(3.57), the Bianchi type IX metric (3.44) becomes the Atiyah-Hitchin metric, with the following substitutions:

$$e^{2f(\eta)} = \frac{2\nu_2\nu_3'\nu_4'}{\pi\nu_2'\nu_3\nu_4}, \tag{3.58}$$

$$e^{2h(\eta)} = \frac{2\nu_2'\nu_3\nu_4'}{\pi\nu_2\nu_3'\nu_4}, \tag{3.59}$$

$$e^{2g(\eta)} = \frac{2\nu_2'\nu_3'\nu_4}{\pi\nu_2\nu_3\nu_4'}, \tag{3.60}$$

where the prime indicates the derivative with respect to the argument η.

In these equations (3.58)-(3.60), ν's are the Jacobi Theta functions and are given by [58]:

$$\nu_1 = \nu[^1_1](z|\tau), \tag{3.61}$$

$$\nu_2 = \nu[^1_0](z|\tau), \tag{3.62}$$

$$\nu_3 = \nu[^0_0](z|\tau), \tag{3.63}$$

$$\nu_4 = \nu[^0_1](z|\tau), \tag{3.64}$$

and $\nu[^a_b](z|\tau)$ is:

$$\nu[^a_b](z|\tau) = \sum_{n\in\mathcal{Z}} e^{\tau(n-\frac{a}{2})+(2z-\frac{b}{2})}e^{i\pi(n-\frac{a}{2})}, \tag{3.65}$$

where a and b are real numbers and z and τ are two complex variables (for more details about the Jacobi Theta functions refer to [59, 60, 61]).

The following choices for the constants $(\lambda_1, \lambda_2, \lambda_3)$ also lead to the same result as in the first case:

$$(\lambda_1, \lambda_2, \lambda_3) = (1, -1, -1), \tag{3.66}$$

$$(\lambda_1, \lambda_2, \lambda_3) = (-1, 1, -1), \tag{3.67}$$

$$(\lambda_1, \lambda_2, \lambda_3) = (-1, -1, 1), \tag{3.68}$$

by substitutions for the functions as [56]:

$$e^f \to -e^f, e^h \to -e^h, e^g \to -e^g, \tag{3.69}$$

respectively for (3.66), (3.67) and (3.68).

3.3 Atiyah-Hitchin Metric

The Atiyah-Hitchin geometry is used in different contexts such as the supersymmetric background of the string theory, M-branes and supergravity [57, 62, 63, 64]. Four-dimensional Atiyah-Hitchin metric is a hyperkähler metric and has an SO(3) transitive isometry [7]. These properties are highly demanded in different theories by supersymmetries, which makes the Atiyah-Hitchin metric a useful geometry.

Approaching differently to the Atiyah-Hitchin metric by SO(3) invariant form, we can write down the line element as:

$$ds^2_{A.H} = f^2(r)dr^2 + a^2(r)\sigma_1^2 + b^2(r)\sigma_2^2 + c^2(r)\sigma_3^2, \tag{3.70}$$

where σ_i's are given in equations (3.45), (3.46) and (3.47). The Atiyah-Hitchin metric (3.70) satisfies Einstein's equations. This leads to the following relationships for the functions $a(r)$, $b(r)$, $c(r)$ and $f(r)$ [65]:

$$2a' = f\frac{(b-c)^2 - a^2}{bc}, \tag{3.71}$$

$$2b' = f\frac{(c-a)^2 - b^2}{ca}, \tag{3.72}$$

$$2c' = f\frac{(a-b)^2 - c^2}{ab}, \tag{3.73}$$

where the primes indicate differentiation with respect to r.

We solve the differential equations (3.71)-(3.73) in terms of the Elliptic integrals (Atiyah-Hitchin was originally expressed in terms of the elliptic functions [66]) by choosing the function $f(r)$ to be:

$$f(r) = \frac{-b(r)}{r}. \tag{3.74}$$

Hence we find [65]:

$$a(r) = (\frac{r\Upsilon\sin(\gamma)\frac{1}{2}\{(1-\cos(\gamma))r - 2\sin(\gamma)\Upsilon\}}{\Upsilon\sin(\gamma) + r\cos^2(\frac{\gamma}{2})})^{1/2}, \tag{3.75}$$

$$b(r) = (\frac{[\Upsilon\sin(\gamma) - \frac{1-\cos(\gamma)}{2}r]r[-\Upsilon\sin(\gamma) - \frac{1+\cos(\gamma)}{2}r]}{\Upsilon\sin(\gamma)})^{1/2}, \tag{3.76}$$

$$c(r) = -(\frac{r\Upsilon\sin(\gamma)\frac{1}{2}\{(1+\cos(\gamma))r + 2\sin(\gamma)\Upsilon\}}{-\Upsilon\sin(\gamma) + \frac{1-\cos(\gamma)}{2}r})^{1/2}, \tag{3.77}$$

where Υ has the following form:

$$\Upsilon = \frac{2nE\{\sin(\gamma/2)\}}{\sin(\gamma)} - \frac{nK\{\sin(\gamma/2)\}\cos(\gamma/2)}{\sin(\gamma/2)}. \tag{3.78}$$

In equation (3.78), E and K are the elliptic integrals which are as below [65]:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\cos^2\theta}}, \tag{3.79}$$

$$E(K) = \int_0^1 \frac{\sqrt{1-k^2t^2}dt}{\sqrt{1-t^2}} = \int_0^{\pi/2} \sqrt{1-k^2\cos^2\theta d\theta}. \tag{3.80}$$

We pause our discussion for a moment to study the Jacobi elliptic function, which will be used in the traxial Bianchi type IX geometry as well.

Jacobi elliptic functions appear in a variety of problems in physics. By realizing that each inverse trigonometric function is a solution of a definite integral, the trigonometric functions *sin* and *cos* can be redefined in terms of the functional inverse of specific integrals [67]:

$$\arcsin z = \int_0^z \frac{dt}{\sqrt{1-t^2}}, \tag{3.81}$$

$$\arctan z = \int_0^z \frac{dt}{1+t^2}, \tag{3.82}$$

where $|z| \leq 1$. Hence, by reinterpreting the equations, we consider these relations (3.81)-(3.82) to define the inverse trigonometric functions. We define the argument θ as:

$$\theta(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}} = \arcsin z, \tag{3.83}$$

and we find the trigonometric function $\sin\theta = z$ by inverting these integrals. Legendre stated that any integrated expression can be reduced to a linear combination of the elliptic integrals of the first, second and the third kind, if it contains a third or fourth degree polynomial in the denominator of a fraction [67]. We mention the elliptic integrals according to the article [67]:

$$F(z,k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{3.84}$$

$$F(\phi,k) = \int_0^\phi \frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}}, \tag{3.85}$$

$$E(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt, \tag{3.86}$$

$$E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2\phi'}d\phi', \tag{3.87}$$

$$\Pi(z,k,n) = \int_0^z \frac{dt}{(1-n^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{3.88}$$

$$\Pi(\phi,k,n) = \int_0^\phi \frac{d\phi'}{(1-n^2\sin^2\phi')\sqrt{1-k^2\sin^2\phi'}}. \tag{3.89}$$

In equations (3.84)-(3.89), $t = \sin\phi'$, $y = \sin\phi$, n is a real number and $k \in (-1,1)$. Equations (3.84), (3.86) and (3.88) are called Jacobi forms and equations (3.85), (3.87) and (3.89) are known as Legendre's form. The integrals are said to be complete when $\phi = \pi/2$, where we write:

$$F(\frac{\pi}{2},k) \equiv K(k). \tag{3.90}$$

By inverting the elliptic integrals of the first kind, Jacobi and Abel introduced the Jacobi elliptic functions. We use the Jacobi notation as follow:

$$u(\phi) \equiv F(z,k), \tag{3.91}$$

and use the inversion of the Legendre form of the integral (3.85) to find the amplitude of u [67]:

$$\phi(u) = \mathbf{am}u. \tag{3.92}$$

Moreover, we define the *sin*-amplitude Jacobi elliptic function by inverting the Jacobi form (3.84) and introducing $u(z) = F(z,k)$:

$$\mathbf{sn}(u,k) \equiv z = \sin\phi = \sin(\mathbf{am}u). \tag{3.93}$$

Equation (3.93) relates the Legendre and Jacobi forms of the elliptic integrals. We define another Jacobi elliptic function as below:

$$\mathbf{cn}(u,k) \equiv \cos\phi = \cos(\mathbf{am}u), \tag{3.94}$$

which is the *cos*-amplitude. Since the term $\Delta(\phi) \equiv \sqrt{1-k^2\sin\phi}$ repeats in all Legendre form of elliptic integrals, we introduce the delta-amplitude [67]:

$$\mathbf{dn}(u,k) \equiv \sqrt{1-k^2\mathbf{sn}^2 u} = \frac{d(\mathbf{am}u)}{du}. \tag{3.95}$$

A few identities for the Jacobi elliptic functions **sn**, **cn** and **am** are [67]:

$$\mathbf{dn}^2 u + k^2\mathbf{sn}^2 u = 1, \tag{3.96}$$

$$\mathbf{sn}^2 u + \mathbf{cn}^2 u = 1, \tag{3.97}$$

$$\mathbf{cn}^2 u + (1-k^2)\mathbf{sn}^2 u = \mathbf{dn}^2 u. \tag{3.98}$$

The behaviour of the Jacobi elliptic functions **sn**, **cn** and **dn** are shown in the figures 3.1, 3.2 and 3.3, respectively, where we set the constant $k = 0.7$ for the elliptic function **dn**.

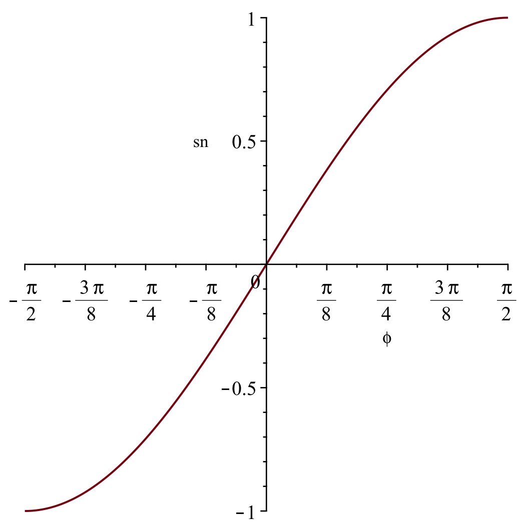

Figure 3.1: The behaviour of the Jacobi elliptic function **sn** with respect to ϕ.

We also show the behaviour of the Jacobi elliptic functions in one graph for $0 \leq \phi \leq \pi$ in figure 3.4.

The Atiyah-Hitchin metric in equation (3.70) reduces to the Euclidean Taub-NUT metric in the limit of $r \to \infty$. The four-dimensional Taub-NUT metric with a negative NUT charge $N = -n$ is be obtained by considering the behaviour of the metric functions $a(r)$, $b(r)$ and $c(r)$ in the limit $r \to \infty$, as below:

$$ds_n^2 = (1-\frac{2n}{r})(dr^2 + r^2 d\Omega^2) + \frac{4n^2}{1-\frac{2n}{r}}(d\psi + \cos\theta d\phi)^2, \tag{3.99}$$

where $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$.

On the other hand, by choosing the metric function $f(r)$ in equation (3.70) to be $f(r) = 4abc$, an analytic solution is found for the metric functions $a(r)$, $b(r)$, $c(r)$. This choice changes the equation (3.70) as below [62]:

$$ds_{A.H}^2 = 16a^2(\eta)b^2(\eta)c^2(\eta)d\eta^2 + a^2(\eta)\sigma_1^2 + b^2(\eta)\sigma_2^2 + c^2(\eta)\sigma_3^2. \tag{3.100}$$

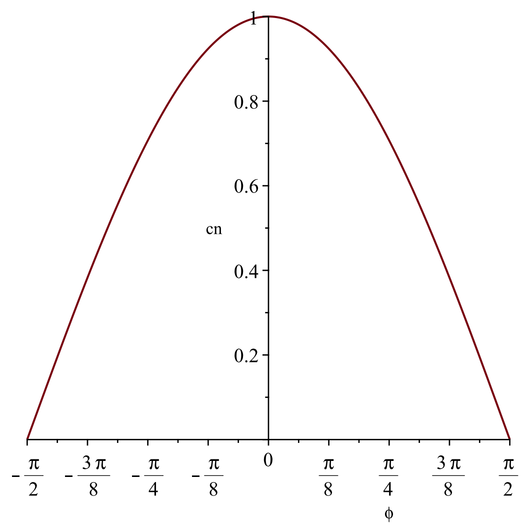

Figure 3.2: The behaviour of the Jacobi elliptic function **cn** with respect to ϕ.

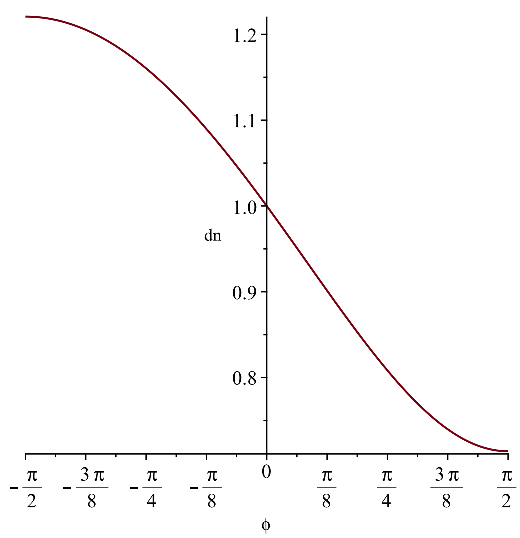

Figure 3.3: The behaviour of the Jacobi elliptic function **dn** with respect to ϕ, where we set the constant $k = 0.7$.

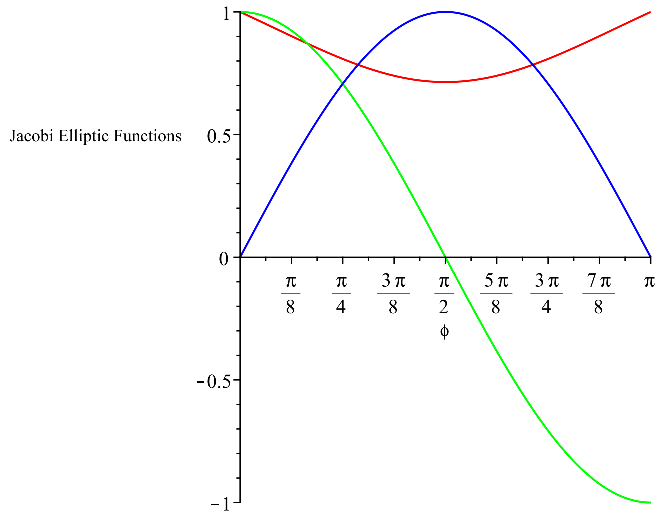

Figure 3.4: The behaviour of the Jacobi elliptic functions **sn**, **cn** and **dn** (which are shown by blue, green and red in the graph, respectively) with respect to ϕ, where we set the constant $k = 0.7$.

The functions $a(\eta)$, $b(\eta)$ and $c(\eta)$ can be expressed in terms of the new functions $\psi_1(\eta)$, $\psi_2(\eta)$ and $\psi_3(\eta)$ as:

$$4a^2(\eta) = \frac{\psi_2\psi_3}{\psi_1}, \tag{3.101}$$

$$4b^2(\eta) = \frac{\psi_3\psi_1}{\psi_2}, \tag{3.102}$$

$$4c^2(\eta) = \frac{\psi_1\psi_2}{\psi_3}. \tag{3.103}$$

By substituting these results in equations (3.71), (3.72) and (3.73), we find the new functions in terms of μ, where $\mu(\nu) = \frac{1}{\pi}K\sqrt{\sin\nu}\sin\frac{\nu}{2}$ [62]:

$$\psi_1 = \frac{-1}{2}(\frac{d\mu^2}{d\nu} + \frac{\mu^2}{\sin\nu}), \tag{3.104}$$

$$\psi_2 = \frac{-1}{2}(\frac{d\mu^2}{d\nu} - \frac{\mu^2\cos\nu}{\sin\nu}), \tag{3.105}$$

$$\psi_3 = \frac{-1}{2}(\frac{d\mu^2}{d\nu} - \frac{\mu^2}{\sin\nu}), \tag{3.106}$$

where the relation between ν and η is $\eta = -\int_\nu^\pi \frac{d\nu}{\mu^2(\nu)}$.

3.4 Triaxial Bianchi Type IX Geometry

The other choice of the constants $(\lambda_1, \lambda_2, \lambda_3)$ is $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$. In this case, the equations (3.55), (3.56) and (3.57) become:

$$2\frac{df}{d\eta} = e^{2h} + e^{2g} - e^{2f}, \tag{3.107}$$

$$2\frac{dh}{d\eta} = e^{2g} + e^{2f} - e^{2h}, \tag{3.108}$$

$$2\frac{dg}{d\eta} = e^{2f} + e^{2h} - e^{2g}. \tag{3.109}$$

These equations can be solved exactly [56]:

$$f(\eta) = \frac{1}{2}\ln(c^2\frac{\mathbf{cn}(c^2\eta, k^2)\mathbf{dn}(c^2\eta, k^2)}{\mathbf{sn}(-c^2\eta, k^2)}), \tag{3.110}$$

$$h(\eta) = \frac{1}{2}\ln(c^2\frac{\mathbf{cn}(c^2\eta, k^2)}{\mathbf{dn}(c^2\eta, k^2)\mathbf{sn}(-c^2\eta, k^2)}), \tag{3.111}$$

$$g(\eta) = \frac{1}{2}\ln(c^2\frac{\mathbf{dn}(c^2\eta, k^2)}{\mathbf{cn}(c^2\eta, k^2)\mathbf{sn}(-c^2\eta, k^2)}). \tag{3.112}$$

The functions **sn**, **cn** and **dn** are the standard Jacobi elliptic functions that we discussed earlier [68]. By changing the coordinate η to $r = \frac{2c}{(\mathbf{sn}(c^2\eta,k^2))^{1/2}}$, the triaxial Bianchi type IX metric is found as [56]:

$$\begin{aligned} ds^2_{tr.BIX} &= \frac{dr^2}{\sqrt{J(r)}} + \frac{r^2}{4}\sqrt{J(r)}\{\frac{(d\psi+\cos\theta d\phi)^2}{1-\frac{a_1^4}{r^4}} \\ &+ \frac{(-\sin\psi d\theta+\cos\psi\sin\theta d\phi)^2}{1-\frac{a_2^4}{r^4}} + \frac{(\cos\psi d\theta+\sin\psi\sin\theta d\phi)^2}{1-\frac{a_3^4}{r^4}}\}. \end{aligned} \tag{3.113}$$

This metric (3.113) can be written in a more compact way in terms of the Maurer-Cartan one-forms:

$$ds^2_{tr.BIX} = \frac{dr^2}{J(r)^{1/2}} + \frac{r^2}{4}J(r)^{1/2}(\frac{\sigma_1^2}{1-\frac{a_1^2}{r^4}} + \frac{\sigma_2^2}{1-\frac{a_2^2}{r^4}} + \frac{\sigma_3^2}{1-\frac{a_3^2}{r^4}}), \tag{3.114}$$

In this equation (3.114), $J(r)$ is:

$$J(r) = (1-\frac{a_1^4}{r^4})(1-\frac{a_2^4}{r^4})(1-\frac{a_3^4}{r^4}), \tag{3.115}$$

where a_1, a_2 and a_3 are three parameters that we choose them to be $a_1 = 0$, $a_2 = 2kc$ and $a_3 = 2c$, where $c > 0$ is a constant. We note that the coordinate r should be $r \geq a_3$, otherwise the metric function $J(r)$ in equation (3.115) becomes negative and therefore the metric (3.114) will contain imaginary parts. Moreover, the periodicity for the angles θ, ψ and ϕ are π, 4π and 2π, respectively [56]. It is worth noting that the four-dimensional Bianchi type IX metric (3.114) is a self-dual and asymptotically locally Euclidean space. The behaviour of the function $J(r)$ in equation (3.115) is shown in figure 3.5.

In our research, we use the triaxial Bianchi type IX geometry as a background to find the exact solutions to the Einstein-Maxwell-dilaton theory. This metric is also used in different theories such as string cosmological model in string theory [55], loop quantum cosmology [69] and M-branes [58].

The Bianchi type IX geometry contains two well-know spaces namely the Eguchi-Hanson type I and type II. By considering $k = 0$ in equation (3.113), the Bianchi type IX metric reduces to the Eguchi-Hanson type I geometry, which is as follow [56]:

$$ds^2_{EHI} = \frac{dr^2}{f(r)} + \frac{r^2}{4}f(r)\{d\theta^2 + \sin^2\theta d\phi^2\} + \frac{r^2}{4f(r)}(d\psi + \cos\theta d\phi)^2, \tag{3.116}$$

where the metric function $f(r)$ is:

$$f(r) = \sqrt{1 - \frac{16c^4}{r^4}}, \tag{3.117}$$

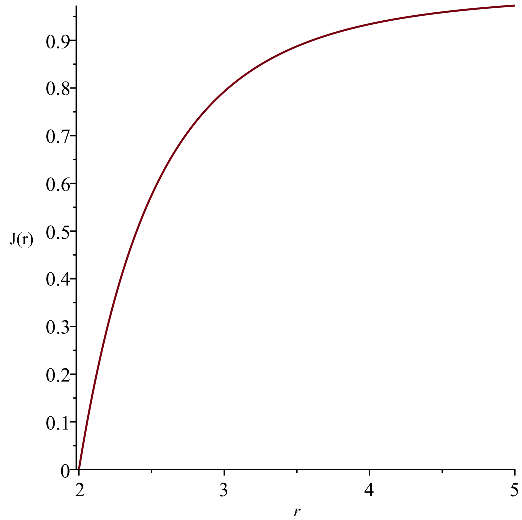

Figure 3.5: The behaviour of the function J(r) with respect to the coordinate r, where we set the constants $k = 0.5$ and $c = 1$.

and c is a constant. The form for the metric function $f(r)$ in (3.117) implies that the radial coordinate r is $r \geq 2c$, where c is a constant.

The Eguchi-Hanson type I space (equation (3.116)) is a self-dual space and asymptotically locally Euclidean with the S^3/Z_2 topology [70]. The metric function $f(r)$ in equation (3.117) is equal to zero when $r = 2c$. This leads to a singularity in the Eguchi-Hanson geometry. This singularity can be removed by limiting the range of the coordinate ψ to $0 \leq \psi \leq 2\pi$. It is noteworthy that the topology of Eguchi-Hanson space near this bolt singularity at $r = 2c$ is $R^2 \times S^2$, where R^2 is the two dimensional Euclidean space and S^2 is the topology of the two-sphere with radius c (in this context).

The Eguchi-Hanson space is an important geometry in different theories such as supergravity (the supergravity solutions based on the Eguchi-Hanson space can be found in [71]), black holes, black hole ring and brane solutions. These solutions based on the Eguchi-Hanson geometry are investigated in [72, 73, 74, 75]. Moreover, the exact solutions to the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type I have been investigated, which we will explain it in detail in other chapters.

Moreover, by choosing $k = 1$, the Bianchi type IX geometry (3.113) reduces to another metric, which is called Eguchi-Hanson type II metric [76]. The metric of Eguchi-Hanson type II is:

$$ds^2_{EH.II} = \frac{dr^2}{f(r)^2} + \frac{r^2 f(r)^2}{4}(d\psi + \cos\theta d\phi)^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.118}$$

where $f(r) = \sqrt{1 - \frac{16c^4}{r^4}}$. The metric in equation (3.118) can be written in a more compact way in terms of the Maurer-Cartan one-forms as:

$$ds^2_{EH.II} = \frac{dr^2}{f(r)^2} + \frac{r^2 f(r)^2}{4}\sigma_1^2 + \frac{r^2}{4}(\sigma_2^2 + \sigma_3^2). \tag{3.119}$$

The denominator of the Ricci scalar and the Kretschmann invariant for the Eguchi-Hanson type II geometry are respectively as below:

$$D_R = r^{10}(16c^4 - r^4), \tag{3.120}$$

$$D_K = r^{12}\sin^4\theta. \tag{3.121}$$

The restriction that we assumed $r > 2c$, removes the singularity at $r = 2c$. This metric (3.119) is singular at $r = 0$ and $\theta = 0$.

We construct the exact solutions to the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II space and compare it to the solutions based on the Bianchi type IX geometry in chapter five.

As we mentioned, the two important metrics Eguchi-Hanson type I and type II are subspaces of a more general geometry, the Bianchi type IX metric, for the extreme cases when the constant k in equation (3.113) is fixed to be 0 and 1, respectively.

Investigating the background geometry that we used in our research, we describe the Einstein-Maxwell-dilaton theory in the next chapter. Finally we find the exact solutions to this theory based on the Bianchi type IX geometry and study the results in chapter five.

4 Einstein-Maxwell-dilaton Theory

The purpose of this section is to construct and study the Einstein-Maxwell-dilaton (EMD) action and its field equations. Hence, we start from the classical field theory to introduce fundamental concepts such as Lagrangian, action and Hamilton's principle. We continue the chapter by constructing the Maxwell and Einstein-Hilbert action and introduce the dilaton field. Finally, we find the action for the Einstein-Maxwell-dilaton theory and derive its field equations and give two examples of the well-known exact solutions to this theory. We note that in this chapter, we put $G = c = 1$, where G is the gravitational constant and c is the speed of light.

4.1 Hamilton's Principle and Classical Field Theory

The fundamental quantity of classical mechanics is the action, S, which is defined as the time integral of the Lagrangian L:

$$S = \int L dt. \tag{4.1}$$

Hamilton's principle states that when a system evolves from one configuration at time t_1 to another at time t_2, it does it in a way that makes the action stationary (it evolves along a path for which S is an extremum). Consider a set of fields Φ^a defined on some general four-dimensional spacetime. L in equation (4.1) is the Lagrangian of the system which is usually a function of the fields Φ^a and their first derivatives. By defining the Lagrangian density $\mathcal{L}$ as [77]:

$$L = \int \mathcal{L} d^3x, \tag{4.2}$$

we rewrite the action (4.1) for this system as the following integral over some four-dimensional region $\mathcal{R}$ of the spacetime:

$$S = \int_{\mathcal{R}} \mathcal{L}(\Phi^a, \partial_\mu \Phi^a) d^4x. \tag{4.3}$$

Consider an infinitesimal variation in the field of the following form:

$$\Phi^a(x) \to \Phi'^a(x) = \Phi^a(x) + \delta\Phi^a(x). \tag{4.4}$$

The variation of the action S under the variation of the field $\Phi^a(x)$ in equation (4.4) is as below:

$$\delta S = \int_{\mathcal{R}} \delta \mathcal{L} d^4x. \tag{4.5}$$

By performing the variation of the Lagrangian density $\delta\mathcal{L}$ with respect to its arguments Φ^a and $\partial_\mu\Phi^a$ we find [19]:

$$\delta S = \int_{\mathcal{R}} d^4x[\frac{\partial\mathcal{L}}{\partial\Phi^a}\delta\Phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\delta(\partial_\mu\Phi^a)], \tag{4.6}$$

where, according to the principle of least action, $\delta S = 0$. As δ commutes with derivatives, the term $\partial_\mu\Phi^a$ in equation (4.6) can be manipulated as below:

$$\delta(\partial_\mu\Phi^a) = \partial_\mu(\delta\Phi^a). \tag{4.7}$$

Substituting (4.7) into (4.6), we find:

$$0 = \delta S = \int_{\mathcal{R}} d^4x[\frac{\partial\mathcal{L}}{\partial\Phi^a}\delta\Phi^a - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)})\delta\Phi^a + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\delta\Phi^a)]. \tag{4.8}$$

The last term in equation (4.8) can be turned into a surface integral over the boundary surface $\partial\mathcal{R}$. Since we assume $\delta\Phi^a$ to be zero over the boundary, this surface term is zero. Hence, we reach to the Euler-Lagrange equation of motion for the system that corresponds to the field element Φ^a:

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}) - \frac{\partial\mathcal{L}}{\partial\Phi^a} = 0. \tag{4.9}$$

As we discussed in chapter 2, the physical quantities should be invariant under Lorentz transformations, i.e., physical entities should be covariant. Hence, the action S should be a Lorentz scalar. Consider a coordinate system x^μ in four-dimensional spacetime. The infinitesimal volume, which is a scalar and is invariant under the Lorentz transformation is denoted as:

$$d^4V = d^4x\sqrt{-g}, \tag{4.10}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. Therefore, we write the action as below [19]:

$$S = \int_{\mathcal{R}} L(\Phi^a, \partial_\mu\Phi^a, \partial_\mu\partial_\nu\Phi^a, ...)\sqrt{-g}d^4x, \tag{4.11}$$

where L is called the field Lagrangian and is usually a function of the fields Φ^a and their first or possibly higher derivatives. Moreover, the field Lagrangian L is related to the Lagrangian density by $\mathcal{L} = L\sqrt{-g}$. Hence, the action S in (4.11) is a Lorentz scalar. It is worth noting that the Lagrangian density $\mathcal{L}$ is a scalar density of weight unity.

As an example of the Lagrangian method, we derive the equations of motion for a real scalar field $\phi(x^\mu)$ that has the following Lagrangian [78]:

$$L = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{4.12}$$

Therefore, the action becomes:

$$S = \int_{\mathcal{R}} d^4x\sqrt{-g}[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2]. \tag{4.13}$$

By calculating $\frac{\partial L}{\partial\phi} = -m^2\phi$ and $\frac{\partial L}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$ and using the Euler-Lagrange equation (4.9) for the real scalar field ϕ, we find the equation of motion:

$$(\partial^\mu\partial_\mu + m^2)\phi = 0, \tag{4.14}$$

which is known as the Klein-Gordon equation. Another important example is the electromagnetic action.

4.2 Maxwell's Action

Constructing the electromagnetic field equations from the Lagrangian method makes us one step closer to the final goal of this chapter, which is finding the Einstein-Maxwell-dilaton action. Electromagnetism can be described in terms of the electromagnetic gauge field A^μ.

The electromagnetic gauge field A_μ has the following gauge symmetry:

$$A'_\mu = A_\mu + \partial_\mu f, \tag{4.15}$$

where f is an arbitrary scalar field. In order to have a set of symmetric equations of motion under the gauge symmetry (4.15), the action of the electromagnetic field must be invariant under this gauge transformation. One possible choice of forming the Lagrangian for the electromagnetic field (which must be Lorentz scalar) is $A_\mu A^\mu$. Although this choice is Lorentz invariant, it leads to some extra terms in the action that violate the gauge symmetry of the equations of motion. Another choice is the electromagnetic field tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.16}$$

which is invariant under the gauge transformation (4.15):

$$F'_{\mu\nu} = \partial_\mu A_\nu + \partial_\mu \partial_\nu f - \partial_\nu A_\mu - \partial_\nu \partial_\mu f = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \tag{4.17}$$

We can form a Lorentz scalar term with respect to the electromagnetic field tensor as $F^{\mu\nu}F_{\mu\nu}$ and construct the Lagrangian as below [19]:

$$L = \frac{-1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \tag{4.18}$$

where μ_0 is the vacuum permeability and the constant $\frac{-1}{4\mu_0}$ is added for further convenience.

By considering the source of the electromagnetic field, we add an interaction term into the Lagrangian of the electromagnetic field (4.18) in terms of the four-current density j^μ:

$$L = \frac{-1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu, \tag{4.19}$$

where the interaction term is denoted by $j^\mu A_\mu$, which is a Lorentz scalar term. The action of the electromagnetic field in presence of the four-current can be constructed with respect to this Lagrangian (4.19) [19]:

$$S = \int_{\mathcal{R}} d^4x \sqrt{-g} [\frac{-1}{4\mu_0} g^{\mu\lambda} g^{\nu\gamma} (\partial_\lambda A_\gamma - \partial_\gamma A_\lambda)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu]. \tag{4.20}$$

By applying the gauge transformation (4.15) on the interaction term $j^\mu A_\mu$ in equation (4.20), and considering that j^μ vanishes on the boundary surface $\partial\mathcal{R}$, we reach to the conservation of the four-current:

$$\nabla_\mu j^\mu = 0, \tag{4.21}$$

which is a consequence of the gauge symmetry.

We find the equation of motion for the electromagnetic field in presence of the source j^μ, by constructing the Euler-Lagrange equation for the action presented in equation (4.20):

$$\frac{\partial L}{\partial A_\nu} = -j^\nu, \tag{4.22}$$

and

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = \frac{-1}{4\mu_0}[(F^{\mu\nu} - F^{\nu\mu}) + (F^{\mu\nu} - F^{\nu\mu})] = \frac{-1}{\mu_0}F^{\mu\nu}. \tag{4.23}$$

Substituting the results in equation (4.9), we find:

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu, \tag{4.24}$$

which is the inhomogeneous Maxwell equation. The other Maxwell's equations can be found from the following identity for the electromagnetic field tensor:

$$\nabla_\gamma F_{\mu\nu} + \nabla_\nu F_{\gamma\mu} + \nabla_\mu F_{\nu\gamma} = 0. \tag{4.25}$$

4.3 Einstein-Hilbert Action

We construct an action for gravitation in the absence of any matter field and find the equations of motion (which is the Einstein field equations) by using the principle of variation. The field generators for gravity are the metric tensors $g_{\mu\nu}$. The best choice for constructing the Lagrangian for gravity is the Ricci scalar R. Not only is this term Lorentz invariant, but also is made of the metric tensor and its derivatives. It is worth noting that R is the only scalar that one can derive from the metric tensor that does not depend on derivatives higher than second order [19]. Hence, the Einstein-Hilbert action will be as bellow:

$$S = \int_{\mathcal{R}} d^4x\sqrt{-g}R, \tag{4.26}$$

where the Lagrangian density is $\mathcal{L} = R\sqrt{-g}$. Using the Euler-Lagrange equation for deriving the equation of motion for this action (4.26) is straight forward but long. Hence, we avoid writing them down and directly show the result of the variation in Einstein-Hilbert action:

$$\delta S = \int_{\mathcal{R}} d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}, \tag{4.27}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar, respectively. As Hilbert's principle states, $\delta S = 0$. Therefore, we find the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{4.28}$$

There are alternative ways to construct the Lagrangian for gravity. One of them is done by Eddington, who chose the Lagrangian to be $L = R_{\mu\nu\gamma\sigma}R^{\gamma\mu\nu\sigma}$ [79].

Now that we found the action of Einstein-Hilbert in vacuum, we consider the existence of the non-gravitational fields and for this purpose, we add a term to the action (4.26) and show it by S_M [19]:

$$S = S_E + S_M, \tag{4.29}$$

where S_E is the action in equation (4.26). Hence:

$$S = \int_{\mathcal{R}} d^4x(\mathcal{L}_E + \mathcal{L}_M), \tag{4.30}$$

where $\mathcal{L}_E$ and $\mathcal{L}_M$ are the Lagrangian that corresponds to the action S_E and S_M, respectively.

By varying the action (4.30) with respect to the metric tensor $g^{\mu\nu}$, we find:

$$G_{\mu\nu}\sqrt{-g} + \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} = 0, \tag{4.31}$$

where the term $(G_{\mu\nu}\sqrt{-g})$ is the variation of $\mathcal{L}_E$ with respect to $g^{\mu\nu}$ that we found in (4.27) and $G_{\mu\nu}$ is the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

If we define the energy-momentum tensor of the extra fields (non-gravitational) as:

$$T_{\mu\nu} = \frac{2\delta\mathcal{L}_M}{\sqrt{-g}\delta g^{\mu\nu}}, \tag{4.32}$$

and substitute it in equation (4.31), we find:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}, \tag{4.33}$$

which is the full Einstein equation that we found earlier [19]. In this equation (4.33), κ is a constant and is equal to $8\pi G/c^4$ (we keep G and c to indicate the dependence of κ on them). The energy-momentum tensor that we defined in (4.32) is symmetric and satisfy the equation $\nabla_\mu T^{\mu\nu} = 0$, as required. The variation of the action S_M with respect to $g^{\mu\nu}$ can be written as:

$$\delta S_M = \int_{\mathcal{R}} d^4x \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \int_{\mathcal{R}} d^4x\sqrt{-g}\frac{-1}{2}T^{\mu\nu}\delta g_{\mu\nu}. \tag{4.34}$$

As an example, consider the action of the electromagnetic field in the absence of the source:

$$S_{em} = \int_{\mathcal{R}} d^4x\sqrt{-g}\frac{1}{-4\mu_0}F^{\mu\nu}F_{\mu\nu}. \tag{4.35}$$

Varying the action of the electromagnetic field (4.35) by $g^{\mu\nu}$, we find [19]:

$$\delta S_{em} = \frac{-1}{4\mu_0}\int_{\mathcal{R}} d^4x\sqrt{-g}(F_{\mu\gamma}F_\nu{}^\gamma - \frac{1}{2}g_{\mu\nu}F_{\gamma\sigma}F^{\gamma\sigma})\delta g^{\mu\nu}. \tag{4.36}$$

Therefore, the energy-momentum tensor for the electromagnetic field becomes:

$$T^{(em)}_{\mu\nu} = \frac{-1}{\mu_0}(F_{\mu\gamma}F_\nu{}^\gamma - \frac{1}{4}g_{\mu\nu}F_{\gamma\sigma}F^{\gamma\sigma}). \tag{4.37}$$

We constructed the action for the Einstein and Maxwell's field, separately. Moreover, we discussed the action and the equations of motion for a scalar field from the Lagrangian method. To satisfy the aim of this chapter, we introduce a scalar field which appears from the compactification of the higher dimensions and is called the *dilaton field*.

4.4 Einstein-Maxwell-dilaton Theory

The higher dimensional theories suggest the existence of an extra scalar field, named the dilaton field, which appears due to the compactification of the extra dimensions. The origin of the dilaton field can be found in the advanced theories such as string theory [80], M-theory [81] and the generalized Freund-Rubin compactification [1, 2, 8, 82], which we do not discuss in this thesis due to their length. However, we mention the Kaluza's idea as a theory which suggests the possibility of the existence of higher dimensions. The basic idea of this theory is to postulate an extra spatial dimension to introduce pure gravity in 5 dimension, which manifests itself in four-dimensional spacetime as the electromagnetic, gravitational and a scalar field [83].

Consider an extra spatial dimension as the fifth dimension. The cylinder condition suggests that the derivatives of the parameters with respect to this new dimension vanish or be very small as they are of higher order [3]. By calculating the Christoffel symbol in five-dimension, we find:

$$\Gamma_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\nu,\gamma} - g_{\nu\gamma,\mu} - g_{\gamma\mu,\nu}), \tag{4.38}$$

$$\Gamma_{5\mu\nu} = \frac{1}{2}(g_{5\mu,\nu} - g_{5\nu,\mu}), \tag{4.39}$$

$$\Gamma_{55\mu} = \frac{1}{2}(g_{55,\mu}), \tag{4.40}$$

$$\Gamma_{555} = 0, \tag{4.41}$$

where index 5 indicates the added spatial dimension. Kaluza assumed the following assumptions to relate the metric tensor and the electromagnetic gauge field [3]:

$$\Gamma_{5\mu\nu} = \alpha F_{\mu\nu}, \tag{4.42}$$

$$\Gamma_{\mu\nu 5} = -\alpha \Sigma_{\mu\nu}, \tag{4.43}$$

$$\Gamma_{55\mu} = -\Gamma_{5\mu 5} = g_{,\mu}\,, \tag{4.44}$$

where α is a constant, $F_{\mu\nu}$ is the electromagnetic field strength, g is the determinant of the metric tensor $g_{\mu\nu}$ and $\Sigma_{\mu\nu}$ is related to the electromagnetic gauge field as $\Sigma_{\mu\nu} = A_{\mu,\nu} + A_{\nu,\mu}$. Considering these assumptions, we find the following relationship for the electromagnetic tensor from the Christoffel symbol:

$$F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu} = 0, \tag{4.45}$$

which yields to Maxwell's equation. Moreover, these equations suggest the existence of a scalar field in the four-dimensional spacetime, when the fifth dimension is compactified. This field is shown by ϕ and is called the dilaton field. The metric tensor in five-dimensions $g_{\alpha\beta}$ (where α and $\beta \in \{1, 2, 3, 4, 5\}$) is a 5×5 matrix that has 25 elements. 16 elements of this matrix is occupied with the tensor metric $g_{\mu\nu}$ in four-dimensions and the other 8 elements are occupied by the electromagnetic gauge field. The remaining element indicates the scalar dilaton field, which we show it by ϕ.

So far, we studied and constructed the Einstein-Hilbert action (in presence and absence of matter/non-gravitational field) and the Maxwell action (in presence and absence of the four-current j^μ). Moreover, we studied the origin of the dilaton field, which appears from the compactification of higher dimensions.

Consider a bosonic field that contains Einstein, Maxwell and dilaton terms, where the dilaton field interacts with Maxwell's field and couples to it with a coupling constant. We investigate this action further by considering the presence of the cosmological constant, as a scalar field, where it interacts with the dilaton field with another coupling constant. We write this action in five-dimensional spacetime as [7]:

$$S = \frac{1}{16\pi}\int d^5x\sqrt{-g}[R - \frac{4}{3}\partial_\mu\phi\partial^\mu\phi - e^{-4/3a\phi}F^{\mu\nu}F_{\mu\nu} - e^{4/3b\phi}\Lambda], \tag{4.46}$$

where R is the Ricci scalar, ϕ is the dilaton field and $F^{\mu\nu}$ is the anti-symmetric electromagnetic field tensor. In this action (4.46), the dilaton field is coupled to both the electromagnetic field and the cosmological constant Λ with two different coupling constants a and b, respectively.

The equations of motion for Maxwell field in five-dimensions can be found by varying the action (4.46) with respect to the electromagnetic gauge field A_μ [70]:

$$M_\mu = \nabla^\nu(e^{-4/3a\phi}F_{\mu\nu}) = 0. \tag{4.47}$$

Moreover, we find the Einstein's equations in five-dimensions by varying the action (4.46) with respect to the metric tensor [7]:

$$\varepsilon_{\mu\nu} = R_{\mu\nu} - \frac{2}{3}\Lambda g_{\mu\nu}e^{4/3b\phi} - (F_\mu^\lambda F_{\nu\lambda} - \frac{1}{6}g_{\mu\nu}F^2)e^{-4/3a\phi} - \frac{4}{3}\nabla_\mu\phi\nabla_\nu\phi = 0. \tag{4.48}$$

Finally, we find the equation of motion for the dilaton field by varying the action (4.46) with respect to the dilaton field generator ϕ:

$$D = \nabla^2\phi + \frac{a}{2}e^{-4/3a\phi}F^2 - \frac{b}{2}e^{4/3b\phi}\Lambda = 0. \tag{4.49}$$

These equations (4.47)-(4.49) are highly non-linear and cannot be solved exactly. Hence, in order to find the exact solutions to this theory, further assumptions are required.

4.5 Exact Solutions to the Einstein-Maxwell-dilaton Theory Based on the Four-Dimensional Eguchi-Hanson Type II Geometry

A new class of exact solutions to the five-dimensional Einstein-Maxwell-dilaton theory with the same action in equation (4.46), where the dilaton field is coupled to both the electromagnetic field and the cosmological constant, is found based on the four-dimensional Eguchi-Hanson type II geometry in the paper [70]. We study these cosmological solutions here.

By considering the most general case for the Einstein-Maxwell-dilaton theory, where the coupling constants are non-zero and not equal to each other, we assume the following form for the five-dimensional metric:

$$ds_5^2 = -\frac{1}{H^2(r,\theta)}dt^2 + R^2(t)H(r,\theta)ds^2_{EH.II}, \tag{4.50}$$

where $ds_{EH.II}$ represents the four-dimensional Eguchi-Hanson type II metric, which is as below:

$$ds^2_{EH.II} = \frac{dr^2}{f(r)^2} + \frac{r^2 f(r)^2}{4}(d\psi + \cos\theta d\phi)^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.51}$$

where $f(r) = (1 - \frac{16c^4}{r^4})$ and figure 4.1 shows the behaviour of this function.

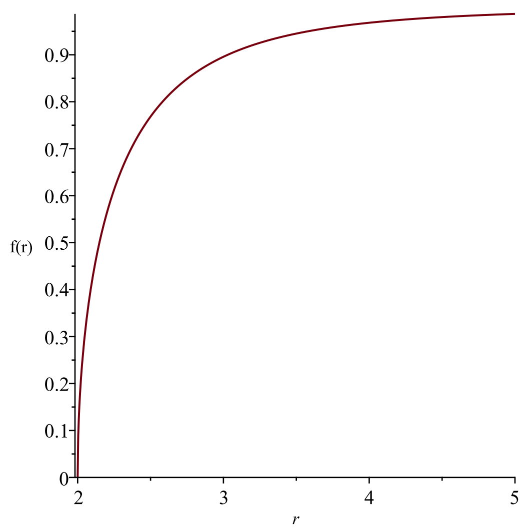

Figure 4.1: The behaviour of the function $f(r)$ with respect to the r coordinate, where we set the constant $c = 1$.

In equation (4.50), $H(r, \theta)$ and $R(t)$ are metric functions that are a function of r and θ coordinates and time coordinate, respectively. These functions will be found analytically through the calculation by satisfying the Einstein, Maxwell and dilaton field equations.

Moreover, we consider ansatzes for the electromagnetic gauge field and dilaton field, respectively:

$$A_t(t, r, \theta) = \alpha R^M(t) H^E(r, \theta), \tag{4.52}$$

$$\phi(t, r, \theta) = -\frac{3}{4a} \ln (H^L(r, \theta) R^W(t)), \tag{4.53}$$

where in the electromagnetic gauge field, α, M and E are constants. As it can be seen, only the t component of the electromagnetic gauge field is considered to be non-zero. The electromagnetic gauge field is a function of time and two spatial coordinates r and θ in this assumption. Also L and W are two arbitrary constants that will be determined from the equations.

The r and θ component of Maxwell's equations are respectively as below:

$$M^r = -\frac{(16c^4 - r^4) H^E R^M (\frac{\partial H}{\partial r})(\frac{dR}{dt}) \alpha E (4Wa + 4Ma + 8a)}{4r^4 a}, \tag{4.54}$$

$$M^\theta = \frac{(16c^4 - r^4) H^E R^M (\frac{\partial H}{\partial \theta})(\frac{dR}{dt}) \alpha E (4Wa + 4Ma + 8a)}{a}, \tag{4.55}$$

where both of them lead to a same relationship for the constants M and W, which is $M + W + 2 = 0$. From the following components of Einstein's equations, we find another relation between the constants:

$$\varepsilon_{tr} = -\frac{3(\frac{dR}{dt})(\frac{\partial H}{\partial r})(LW + 4a^2)}{4a^2 HR}, \tag{4.56}$$

$$\varepsilon_{t\theta} = -\frac{3(\frac{dR}{dt})(\frac{\partial H}{\partial\theta})(LW + 4a^2)}{4a^2HR}, \tag{4.57}$$

which yield to the following equation:

$$LW = -4a^2. \tag{4.58}$$

Moreover, the Einstein equation $\varepsilon_{r\theta}$ reads:

$$\varepsilon_{r\theta} = \frac{(\frac{\partial H}{\partial r})(\frac{\partial H}{\partial\theta})(4R^{W+2M}H^{L+2E+2}\alpha^2E^2a^2 - 3L^2 - 6a^2)}{4a^2H^2}, \tag{4.59}$$

which leads to a few other relations between the constants. Combining the results, the constants in the considered ansatzes are determined as:

$$E = -1 - a^2/2, M = 2, W = -4, \alpha^2 = \frac{3}{a^2+2}, L = a^2. \tag{4.60}$$

From Maxwell's equation M^t, we find the metric function $H(r,\theta)$ as:

$$H(r,\theta) = (1 + \frac{g_+}{r^2 + 4c^2\cos\theta} + \frac{g_-}{r^2 - 4c^2\cos\theta})^{\frac{2}{2+a^2}}. \tag{4.61}$$

We show the behaviour of the metric function $H(r,\theta)$ in figure 4.2, with respect to the coordinates r and θ. We notice from the figure that the metric function $H(r,\theta)$ is regular. Moreover, there are spikes that appear around $r = 0$ and $\theta = 0$.

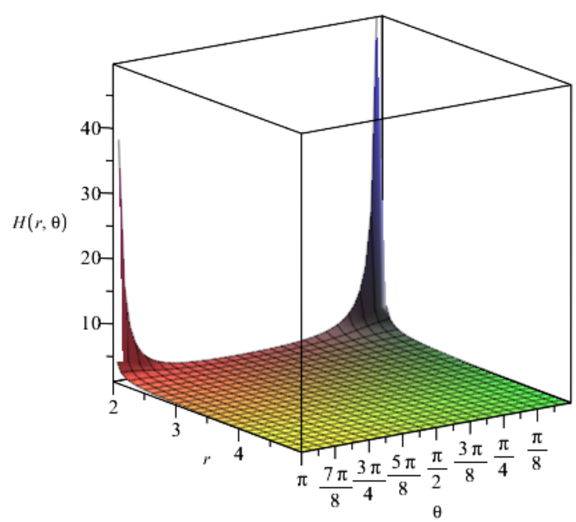

Figure 4.2: The metric function $H(r,\theta)$ as a function of the coordinates r and θ, where the constants are set $g_+ = 2$, $g_- = 3$, $a = 1$ and $c = 1$.

From the other equations, we determine the form of the other metric function $R(t)$ as below:

$$R(t) = (\eta t + \zeta)^{a^2/4}, \tag{4.62}$$

from which we find the cosmological constant:

$$\Lambda = \frac{3}{8}\eta^2a^2(a^2 - 1). \tag{4.63}$$

Equation (4.63) indicates that the cosmological constant can be positive, negative or zero, depending on the choice of the coupling constant a. More importantly, we find a relation between the coupling constants:

$$ab = -2. \tag{4.64}$$

These results satisfy all of the equations of Einstein, Maxwell and dilaton.

From equation (4.64) it is obvious that the coupling constants cannot be equal to each other. Hence we consider a new set of ansatzes for the five-dimensional metric, electromagnetic gauge field and dilaton field:

$$ds_5^2 = -\frac{1}{H^2(t,r,\theta)}dt^2 + R^2(t)H(t,r,\theta)ds^2_{EH.II}, \tag{4.65}$$

$$A_t(t,r,\theta) = \alpha R^M(t)H^E(t,r,\theta), \tag{4.66}$$

$$\phi(t,r,\theta) = -\frac{3}{4a}\ln\left(H^L(t,r,\theta)R^W(t)\right), \tag{4.67}$$

where the four-dimensional metric $ds^2_{EH.II}$ is given in equation (4.51) and M, E, L and W are constants. Using a similar method as we used for the case where the coupling constants were not equal to each other, we find these constants:

$$L = a^2, M = -a^2, W = 2a^2, E = -1 - a^2/2. \tag{4.68}$$

According to these results (4.68), we find the metric functions $H(t,r,\theta)$ as below [70]:

$$H(t,r,\theta) = (R^{a^2+2}(t) + K(r,\theta))^{\frac{2}{a^2+2}}R^{-2}(t), \tag{4.69}$$

where the metric function $R(t)$ is:

$$R(t) = (\epsilon t + \mu)^{1/a^2}, \tag{4.70}$$

and the function $K(r,\theta)$ has the following form:

$$K(r,\theta) = 1 + \frac{k_+}{r^2 + 4c^2\cos\theta} + \frac{k_-}{r^2 - 4c^2\cos\theta}. \tag{4.71}$$

In equations (4.70) and (4.71), ϵ, μ and $k_\pm$ are arbitrary constants.

Moreover, the cosmological constant is found to be [70]:

$$\Lambda = \frac{3}{2}\epsilon^2\frac{4-a^2}{a^4}, \tag{4.72}$$

which again can be positive, negative or zero based on the coupling constant a.

4.6 Exact Solutions to the Einstein-Maxwell-dilaton Theory Based on the Four-Dimensional Taub-NUT Geometry

Another Exact solution to the Einstein-Maxwell-dilaton theory is found based on the four-dimensional Taub-NUT geometry [7]. Starting from the Einstein-Maxwell-dilaton action (4.46), where the dilaton field is coupled to the electromagnetic field as well as the cosmological constant, we find the Maxwell, Einstein and dilaton equations of motion by varying the action (4.46) with respect to the electromagnetic gauge field, the metric tensor and the dilaton field, respectively. The results are already shown in equations (4.47), (4.48) and (4.49).

We assume the following ansatz for the five-dimensional metric [7]:

$$ds_5^2 = \frac{1}{H^2(r)} dt^2 + R^2(t) H(r) ds_{TN}^2, \tag{4.73}$$

where ds_{TN}^2 is the four-dimensional multi-center Taub-NUT space. We consider the one-center Taub-NUT space:

$$ds_{TN}^2 = (1 + \frac{n}{r})(dr^2 + r^2 d\Omega^2) + \frac{1}{1 + \frac{n}{r}}(d\psi + \cos\theta d\phi)^2, \tag{4.74}$$

where it is written in terms of a positive NUT charge n. The electromagnetic gauge field ansatz and the dilaton field ansatz are respectively assumed to be:

$$A_t(t, r) = \xi R(t)^2 (F(r) - \chi), \tag{4.75}$$

$$\phi(t, r) = -\frac{3}{4a} \ln R(t)^\mu H(r)^\nu, \tag{4.76}$$

where $F(r)$ is a function of coordinates r and ξ, and χ, μ and ν are constants. The Maxwell's component M^t gives the following form for the function $F(r)$ that is used in electromagnetic gauge ansatz (4.75) [7]:

$$F(r) = F_1 + h \int \frac{dr}{r^2 H^{\nu+2}(r)}, \tag{4.77}$$

where F_1 and h are constants. From the Maxwell's component M^r and Einstein's component ε_{tr}, we find the constants μ and ν to be $\mu = -4$ and $\nu = a^2$. Substituting these results, the dilaton field (4.76) becomes:

$$\phi(t, r) = -\frac{3}{4a} \ln \frac{H^{a^2}(r)}{R^4(t)}. \tag{4.78}$$

From the other equations, we find the metric functions $H(r)$ and $R(t)$ [7]:

$$H(r) = (1 + \frac{h}{r})^{\frac{2}{2+a^2}}, \tag{4.79}$$

$$R(t) = R_0 t^{a^2/4}, \tag{4.80}$$

where R_0 is a constant. Substituting (4.79) in (4.77), the function $F(r)$ reads $F(r) = \frac{r}{r+h}$. Moreover, the cosmological constant becomes:

$$\Lambda = \frac{3a^2}{8}(a^2 - 1). \tag{4.81}$$

The cosmological constant can be positive, negative and zero based on the choice of the coupling constant a. The same relation between the coupling constants exists:

$$ab = -2. \tag{4.82}$$

From equation (4.82) it can be seen that in order to have equal coupling constants in the theory, new set of ansatzes are required as the case $a = b$ does not satisfy the condition $ab = -2$.

Hence, consider the case where the coupling constants are equal to each other and non-zero. The following ansatz for the five-dimensional metric is assumed [7]:

$$ds_5^2 = -\frac{1}{H^2(t,r)}dt^2 + R^2(t)H(t,r)ds_n^2, \tag{4.83}$$

where ds_n is given in (3.99). The only difference between the five-dimensional ansatz in equations (4.74) and (4.83) is that the metric function $H(t,r)$ depends on the time coordinate as well as the radial coordinate r in the latter one, while in the former one, it only depends on the radial coordinate. Moreover, the assumed ansatz for the electromagnetic gauge field is:

$$A_t(t,r) = \frac{\xi}{R^{a^2}(t)}(F(t,r) - \chi), \tag{4.84}$$

where ξ and χ are constants. The Maxwell's component M^t yields to a solution for $F(t,r)$:

$$F(t,r) = F_1(t) + F_2(t)\int \frac{dr}{r^2 H^{2+\nu}(t,r)}, \tag{4.85}$$

where ν is a constant that according to the Maxwell component M^r, is equal to $\nu = a^2$. By using a similar method that we used for the case where the coupling constants $a \neq b$, we find the following results [7]:

$$\phi(t,r) = -\frac{3}{4a}\ln R^{2a^2}(t)H^{a^2}(t,r), \tag{4.86}$$

$$H(t,r) = (1 + \frac{h}{rR^{a^2+2}(t)})^{\frac{2}{a^2+2}}, \tag{4.87}$$

$$R(t) = \alpha t^{\frac{1}{a^2}}, \tag{4.88}$$

where h and α are constants. We find the cosmological constant as below:

$$\Lambda = \frac{3(4-a^2)}{2a^4}. \tag{4.89}$$

We find the exact solutions to the Einstein-Maxwell-dilaton theory based on a more general geometry, named Bianchi type IX, in presence of the cosmological constant and derive the equations of motion in the next chapter.

5 New Classes of Exact Solutions to the Einstein-Maxwell-dilaton Theory on a Four-Dimensional Bianchi Type IX Geometry

In this chapter, we find the new class of exact solutions to the Einstein-Maxwell-dilaton theory, where the dilaton field is coupled to the electromagnetic field as well as the cosmological constant, with two different coupling constants. First we consider the most general case, where the coupling constants are not equal to each other. The other cases that we consider are where the coupling constants are equal to each other and are non-zero, and where they are both equal to zero. The latter one leads to the Einstein-Maxwell theory. Moreover, we find a new combined solutions to the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II geometry. We discuss the properties of our results in each case.

5.1 The Exact Solutions to the Einstein-Maxwell-dilaton Theory, Based on the Bianchi Type IX Geometry, with Two Different Coupling Constants a and b

We consider the cosmological Einstein-Maxwell-dilaton (EMD) theory, where the dilaton field is coupled to both the electromagnetic field and the cosmological constant, with two different coupling constants. The action for the theory in $N+1$ dimensions, in presence of the cosmological constant Λ, is given by [84]:

$$S = \frac{1}{16\pi}\int d^{N+1}x\sqrt{-g}\{R - \frac{4}{N-1}(\nabla\phi)^2 - e^{4/(N-1)a\phi}F^2 - e^{4/(N-1)b\phi}\Lambda\}. \tag{5.1}$$

In (5.1), the electromagnetic field strength $F_{\mu\nu}$ and the cosmological constant interact with the dilaton field ϕ where a and b are two arbitrary coupling constants. This action (5.1) contains the Einstein field, Maxwell field and dilaton field, which we discussed them separately in the previous chapter.

We also note that in (5.1), R represents the curvature scalar and $g = det[g_{\mu\nu}]$ [19, 22].

Variation of the action (5.1) with respect to the electromagnetic gauge field A_μ, leads to the Maxwell field equations in $N+1$ dimensions [84]:

$$M_\mu \equiv \nabla^\nu(e^{-4/(N-1)a\phi}F_{\mu\nu}) = 0. \tag{5.2}$$

By varying the Einstein-Maxwell-dilaton action (5.1) with respect to the metric tensor $g_{\mu\nu}$, we find the

Einstein field equations in $N+1$ dimensions [84]:

$$\begin{aligned}
\varepsilon_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{4}{N-1}\{\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\} \\
&- e^{\frac{-4a\phi}{N-1}}\{2F_{\mu\lambda}F_\nu^\lambda - \frac{1}{2}g_{\mu\nu}F^2\} - \frac{1}{2}e^{\frac{4b\phi}{N-1}}g_{\mu\nu}\Lambda = 0.
\end{aligned} \tag{5.3}$$

Moreover, by varying the action (5.1) with respect to the dilaton field, we find the dilaton field equation, in $N+1$ dimension, as given by:

$$D \equiv \nabla^2\phi - \frac{b}{2}e^{4/(N-1)b\phi}\Lambda + \frac{a}{2}e^{-4/(N-1)a\phi}F^2 = 0. \tag{5.4}$$

We embed the four-dimensional Bianchi type IX metric, which is a self-dual and asymptotically locally Euclidean space, in the five dimensional metric of the theory. As we discussed, the triaxial Bianchi type IX metric is given by the equation (3.113):

$$\begin{aligned}
ds^2_{tr.BIX} &= \frac{dr^2}{\sqrt{J(r)}} + \frac{r^2}{4}\sqrt{J(r)}\{\frac{(d\psi + \cos\theta d\phi)^2}{1-\frac{a_1^4}{r^4}} \\
&+ \frac{(-\sin\psi d\theta + \cos\psi\sin\theta d\phi)^2}{1-\frac{a_2^4}{r^4}} + \frac{(\cos\psi d\theta + \sin\psi\sin\theta d\phi)^2}{1-\frac{a_3^4}{r^4}}\}.
\end{aligned} \tag{5.5}$$

where, the metric function $J(r)$ is:

$$J(r) = (1-\frac{a_1^4}{r^4})(1-\frac{a_2^4}{r^4})(1-\frac{a_3^4}{r^4}). \tag{5.6}$$

It is worth noting that in (5.6), a_1, a_2 and a_3 are three parameters that we choose to be $a_1 = 0$, $a_2 = 2kc$ and $a_3 = 2c$, where $c > 0$ is a constant and k belongs to the interval $0 \leq k \leq 1$ [56]. We note the coordinate r should be $r \geq a_3$, otherwise the metric function $J(r)$ in equation (5.6) would be negative and therefore the metric (5.5) would contain imaginary parts. Moreover, the periodicity for the angles θ, ψ and ϕ are π, 4π and 2π, respectively.

The action for the Einstein-Maxwell-dilaton theory, where the dilaton field is coupled to the Maxwell field (by the coupling constant a) and to the cosmological constant (by the coupling constant b), can be written in five-dimensions (4+1 dimensions) as:

$$S = \frac{1}{16\pi}\int d^5x\sqrt{-g}\{R - \frac{4}{3}(\nabla\phi)^2 - e^{-4/3a\phi}F^2 - e^{4/3b\phi}\Lambda\}, \tag{5.7}$$

where R is the Ricci scalar, ϕ is the massless dilaton field, F is the electromagnetic tensor and Λ is the cosmological constant

In this section, we consider the general case where the coupling constants are non-zero and $a \neq b$. Such a situation has applications in the generalized Freund-Rubin compactification [8].

By varying the action (5.7) with respect to the electromagnetic gauge field A_μ (or set $N = 4$ in equation (5.2)), we find the Maxwell field equations in five-dimension:

$$M_\mu = \nabla^\nu(e^{-4/3a\phi}F_{\mu\nu}) = 0. \tag{5.8}$$

Also dilaton field equation can be found in five-dimensions by varying the action with respect to the dilaton field (or set $N = 4$ in equation (5.4)), and is given by:

$$D = \nabla^2\phi + \frac{a}{2}e^{-4/3a\phi}F^2 - \frac{b}{2}e^{4/3b\phi}\Lambda = 0. \tag{5.9}$$

Moreover, the Einstein field equations can be obtained by the variation of the action (5.7) with respect to the metric tensor $g_{\mu\nu}$ [7]:

$$\varepsilon_{\mu\nu} = R_{\mu\nu} - \frac{2}{3}\Lambda g_{\mu\nu}e^{4/3b\phi} - (F_\mu^\lambda F_{\nu\lambda} - \frac{1}{6}g_{\mu\nu}F^2)e^{-4/3a\phi} - \frac{4}{3}\nabla_\mu\phi\nabla_\nu\phi = 0. \tag{5.10}$$

We consider an ansatz for the five-dimensional metric as:

$$ds_5^2 = -\frac{1}{H^2(r,\theta)}dt^2 + R^2(t)H(r,\theta)ds^2_{B.IX}, \tag{5.11}$$

where $ds^2_{B.IX}$ is the four-dimensional Bianchi type IX metric (5.5), and $H(r,\theta)$ and $R(t)$ are two metric functions that we determine them later by solving the field equations.

We consider an ansatz for the electromagnetic gauge field, as given by:

$$A_t(t,r,\theta) = \alpha R^M(t)H^E(r,\theta), \tag{5.12}$$

where α, M and E are constants, so the only non-zero component of the electromagnetic gauge field is the t component. The electromagnetic gauge field is a function of time and spatial coordinates r and θ. According to the ansatz (5.12) for the electromagnetic gauge field, we find the electromagnetic field strength $F_{\mu\nu}$ as follow:

$$F_{\mu\nu} = \begin{bmatrix} 0 & \alpha H^E E(\frac{\partial H}{\partial r})R^M/H & \alpha H^E E(\frac{\partial H}{\partial \theta})R^M/H & 0 & 0 \\ -\alpha H^E E(\frac{\partial H}{\partial r})R^M/H & 0 & 0 & 0 & 0 \\ -\alpha H^E E(\frac{\partial H}{\partial \theta})R^M/H & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.13}$$

Finally, we consider an ansatz for the dependence of the dilaton field on the metric functions $H(r,\theta)$ and $R(t)$, as given by:

$$\phi(t,r,\theta) = -\frac{3}{4a}\ln(H^L(r,\theta)R^W(t)), \tag{5.14}$$

where L and W are constants.

Considering the ansatzes (5.11), (5.12) and (5.14), the Maxwell's equations for the components M^r and M^ϕ become:

$$\begin{aligned} M^r &= \frac{-1}{4a}\sqrt{\frac{(2ck-r)(2ck+r)(4c^2k^2+r^2)(2c-r)(2c+r)(4c^2+r^2)}{r^8}} \\ &\times H^E(r,\theta)(\frac{\partial H}{\partial r})R^M(t)E\alpha(\frac{\partial R}{\partial t})(4aW+4Ma+8a), \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} M^\phi &= \frac{-64}{a}r^2\sqrt{\frac{r^8 - 16c^4(k^4+1)r^4 + 256c^8k^4}{r^8}}\cos\psi\sin\psi \\ &\times c^4\alpha H^E E(\frac{\partial H}{\partial \theta})R^M(\frac{\partial R}{\partial t})(4aW+4Ma+8a), \end{aligned} \tag{5.16}$$

where c and k are the constants in the Bianchi type IX geometry (5.5). Both equations (5.15) and (5.16) yield to the following relation for the constants M and W:

$$M + W = -2. \tag{5.17}$$

Also, the Maxwell's components M^{ψ} and M^{θ} become:

$$\begin{aligned} M^{\psi} &= \frac{-16}{a} r^2 \sqrt{\frac{r^8 - 16c^4(k^4+1)r^4 + 256c^8k^4}{r^8}} \cos\psi \sin\psi \\ &\times \; c^4 \cos\theta \alpha H^E E(\frac{\partial H}{\partial \theta}) R^M (\frac{\partial R}{\partial t})(aW + Ma + 2a)(k^4 - 1), \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} M^{\theta} &= \frac{-1}{\alpha}[(-16\sin^2\psi c^4k^4 - 16\cos^2\psi c^4 + r^4)(aW + Ma + 2a) \\ &\times \; (\alpha H^E E(\frac{\partial H}{\partial \theta}) R^M (\frac{dR}{dt})]. \end{aligned} \tag{5.19}$$

Equations (5.18) and (5.19) lead to the same relation between the constants M and W (5.17).

Moreover, from the Einstein's component ε_{tr}:

$$\varepsilon_{tr} = \frac{-3}{4} \frac{(\frac{\partial R}{\partial t})(\frac{\partial H}{\partial r})(LW + 4a^2)}{H(r,\theta)a^2 R(t)}, \tag{5.20}$$

we find the following constraint on the constants L and W:

$$LW + 4a^2 = 0. \tag{5.21}$$

The other Einstein component $\varepsilon_{r\theta}$ reads:

$$\varepsilon_{r\theta} = 4\alpha^2 (H^E(r,\theta))^2 E^2 (R^M(t))^2 H^2(r,\theta) a^2 R^W(t) H^L(r,\theta) - 3L^2 - 6a^2 = 0. \tag{5.22}$$

We find that the constants M, E and α in (5.12), are given by:

$$M = 2, \tag{5.23}$$

$$E = -1 - \frac{a^2}{2}, \tag{5.24}$$

$$\alpha^2 = \frac{3}{a^2+2}. \tag{5.25}$$

Moreover, we find that the constants L and W in (5.14) are given by:

$$L = a^2, \tag{5.26}$$

$$W = -4. \tag{5.27}$$

Hence, equation (5.22) satisfies as $\varepsilon_{r\theta} = 0$.

We present the Maxwell's equation M_t in the Appendix. The analysis of the equation leads to the solutions for the metric function $H(r,\theta)$, which are given by:

$$H(r,\theta) = (j_+ r^2 \cos\theta + j_-)^{\frac{2}{a^2+2}}, \tag{5.28}$$

where j_+ and j_- are two constants. We notice from equation (5.11) that the metric function $H(r,\theta)$ should be a real and positive function. Hence, we should impose the following condition:

$$a^2 + 2 = 2n + 1, \tag{5.29}$$

where $n \in N$. Equation (5.29) leads to an extra constraint on the coupling constant a. Therefore, a can be 1, $\sqrt{3}$, $\sqrt{5}$, ... It is worth noting that by choosing the coupling constant $a = 1$ and $a = \sqrt{3}$, the Einstein-Maxwell-dilaton action (5.1) leads to the low-energy effective action for heterotic string theory and Kaluza-Klein reduction of 5-dimension Einstein gravity, respectively [85]. The behaviour of the metric function $H(r,\theta)$ with respect to r and θ is shown in figure 5.1, where we set $j_+ = 0.5$, $j_- = 15$ and $a = 1$. We notice from the figure that the metric function $H(r,\theta)$ behaves smoothly.

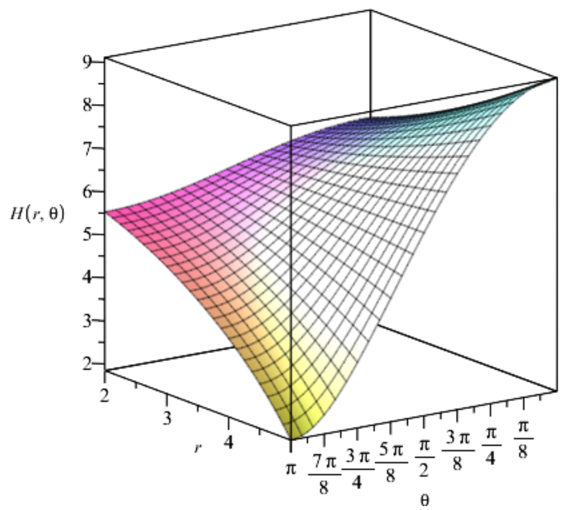

Figure 5.1: The metric function $H(r,\theta)$ as a function of the coordinates r and θ, where we set $j_+ = 0.5$, $j_- = 15$, $c = 1$ and $a = 1$.

We then find the solutions for $R(t)$ and the cosmological constant Λ, considering the other Einstein and Maxwell field equations, though these equations are lengthy, and so we do not present them explicitly here. We find the solutions for $R(t)$ and Λ as:

$$R(t) = (\eta t + \vartheta)^{a^2/4}, \tag{5.30}$$

$$\Lambda = \frac{3}{8}\eta^2 a^2 (a^2 - 1), \tag{5.31}$$

where η and ϑ are arbitrary constants. We also find a relation between the coupling constants a and b, as given by:

$$ab = -2. \tag{5.32}$$

The behaviour of the metric function $R(t)$ with respect to the time coordinate t is shown in figure 5.2.

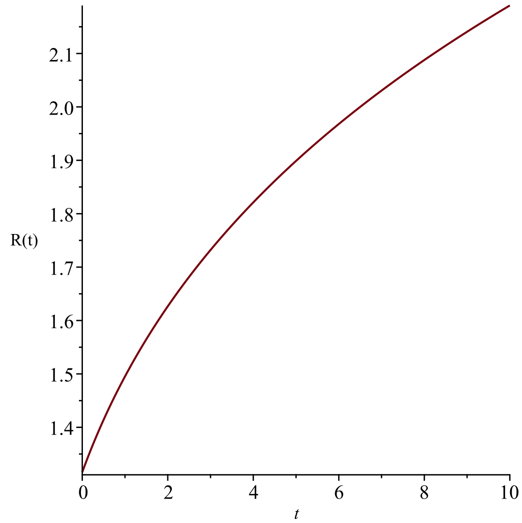

Figure 5.2: The behaviour of the metric function $R(t)$ with respect to the time coordinate t, where we set the constants $\eta = 2$, $\vartheta = 2$ and $a = 1$.

Furnished with all the metric functions in (5.11), we find that the Ricci scalar and also the Kretschmann invariant of the metric (5.11) are divergent on the hypersurfaces $H(r,\theta) = 0$ and $R(t) = 0$. The denominator of the Ricci scalar and the Kretschmann invariant is:

$$D_R = 2r^6 H^3 R^2 \sin^2\theta[(16c^4 - r^4)(16c^4k^4 - r^4)]^{3/2}, \tag{5.33}$$

$$D_K = 4r^{10} H^6 R^4 \sin^4\theta[(16c^4 - r^4)(16c^4k^4 - r^4)]^{15/2}. \tag{5.34}$$

We should note the same type of singularities exists for the supergravity solutions (in more than four-dimensions) [58], that can be avoided with considering more spatial coordinates in the metric functions. Moreover, the singularity at $R(t) = 0$ can be removed by restricting the constants η and ϑ in the metric function (5.30), by $\eta \geq 0$ and $\vartheta > 0$.

We can rewrite the action (5.7) as:

$$S = \frac{1}{16\pi}\int d^5x\sqrt{-g}\{R - \frac{4}{3}(\nabla\phi)^2 - e^{\frac{-4a}{3}\phi}F^2 - e^{\frac{-8}{3a}\phi}\Lambda\}. \tag{5.35}$$

As the coupling constant a increases, the strength of the interaction between the dilaton and the cosmological constant increases, while the strength of the interaction between the dilaton and the electromagnetic field decreases.

According to the equation (5.31), the cosmological constant Λ can be negative, positive or zero depending on the coupling constant a. As it is known in asymptotically AdS/dS spacetimes, the near boundary or even the deep events are holographically dual to the conformal field theory [86]. In asymptotically dS spacetimes, we can interpret the holography in terms of the renormalization group flows in the context of the c-theorem [65]. According to the c-theorem, the renormalization group flows to the infrared in any contracting dS spacetime, and to the ultraviolet for any expanding dS spacetime [87, 88]. By plotting the c-function, which is given by [7]:

$$c \sim \frac{1}{{G_{tt}}^{3/2}}, \tag{5.36}$$

with respect to the time coordinate, we infer the dS spacetime as an expanding or contracting. We show the behaviour of the c-function for the five-dimensional spacetime (5.11), where the coupling constant $a > 1$, in figure 5.3. According to this figure, our five-dimensional spacetime expands by time.

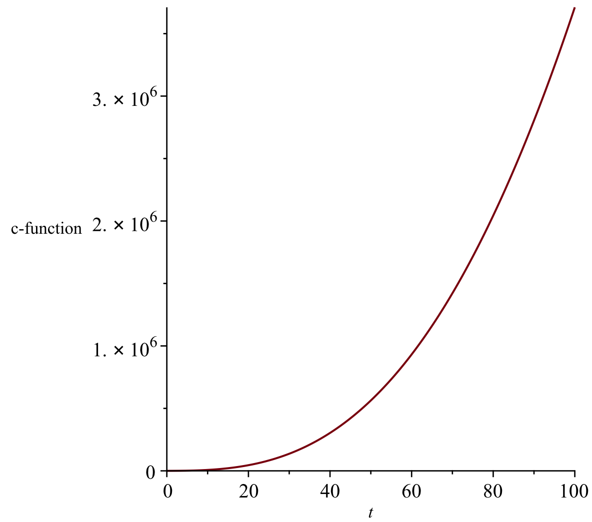

Figure 5.3: The c-function for the five-dimensional spacetime, where the coupling constants are not equal.

We also note that our solutions to the Einstein-Maxwell-dilaton theory based on the Bianchi type IX metric (which contains an arbitrary constant k where $0 \leq k \leq 1$), are completely independent of the constant k.

We show the behaviour of the dilaton field with respect to the coordinates r and θ, in figure 5.4 for two different time slices. In this figure, the upper hypersurface corresponds to the later time. Moreover, figures 5.5 and 5.6 show the components of the electric field with respect to the coordinates r and θ.

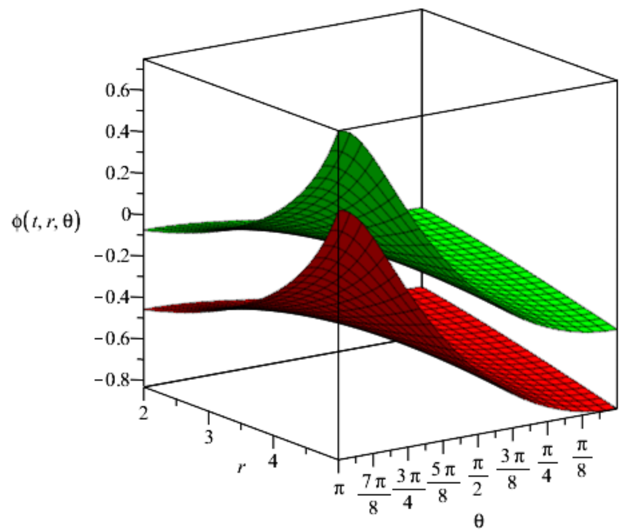

Figure 5.4: The dilaton field $\phi(t, r, \theta)$ as a function of the coordinates r and θ for two different time slices, $t = 1$ and $t = 3$, which are the lower and the upper surface, respectively. We set the constants $j_+ = 0.5$, $j_- = 15$, $a = 1$, $\eta = 1$ and $\vartheta = 2$.

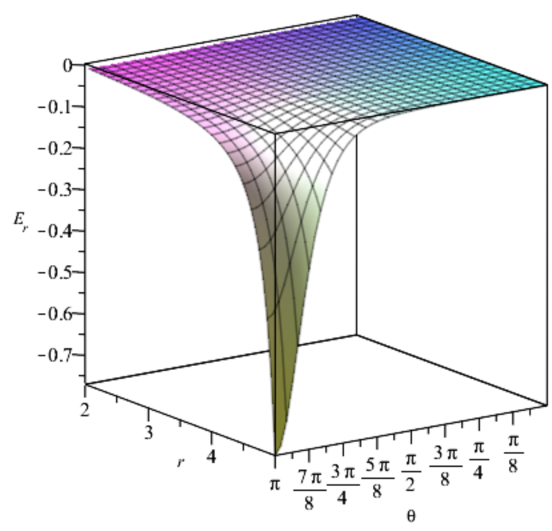

Figure 5.5: The r-component of the electric field as a function of the coordinates r and θ for $t = 1$. We set the constants as $j_+ = 0.5$, $j_- = 15$, $a = 1$, $\eta = 1$, $\vartheta = 2$, $c = 1$.

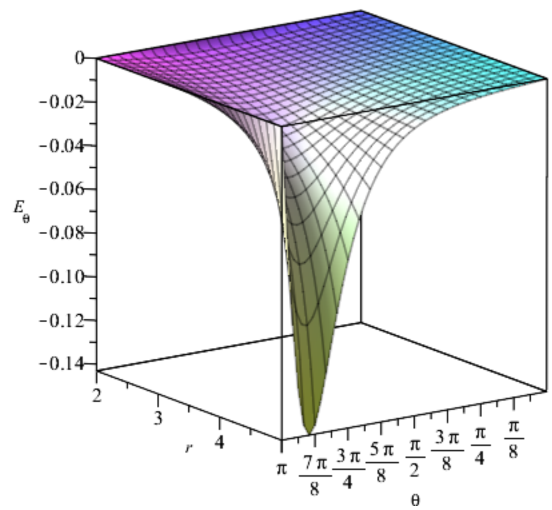

Figure 5.6: The θ-component of the electric field as a function of the coordinates r and θ for $t = 1$. We set the constants as $j_+ = 0.5$, $j_- = 15$, $a = 1$, $\eta = 1$, $\vartheta = 2$, $c = 1$.

5.2 Exact Solutions to the Einstein-Maxwell-dilaton Theory Based on the Bianchi Type IX Geometry, Where the Non-zero Coupling Constants are Equal $a = b \neq 0$

In this section, we find the exact solutions to the five-dimensional Einstein-Maxwell-dilaton theory, where the coupling constants a and b are equal to each other. We need to consider other ansatzes for the metric, the electromagnetic field and the dilaton field, as the ansatzes (5.11), (5.12) and (5.14) lead to $ab = -2$ for the coupling constants, which cannot be satisfied for $a = b$. In what follows, we first consider the case where the coupling constants are equal to each other and non-zero. Second, we consider the case where the coupling constants are both equal to zero. The second case provides solutions to the Einstein-Maxwell theory in the presence of the cosmological constant.

We consider the five-dimensional metric, the electromagnetic field and the dilaton field as:

$$ds_5^2 = -\frac{1}{H^2(t,r,\theta)}dt^2 + R^2(t)H(t,r,\theta)ds^2_{B.IX}, \tag{5.37}$$

$$A_t(t,r,\theta) = \alpha R^M(t)H^E(t,r,\theta), \tag{5.38}$$

$$\phi(t,r,\theta) = -\frac{3}{4a}\ln\left(H^L(t,r,\theta)R^W(t)\right), \tag{5.39}$$

where M, E, L and W are constants and $ds^2_{B.IX}$ is given in (5.5). Though (5.37)-(5.39) are similar to (5.11)-(5.14), however the metric function $H(t,r,\theta)$ depends on time coordinate, beside its dependence on the coordinates r and θ.

By considering the Maxwell and the Einstein field equations, we find the constants M and E in electromagnetic field ansatz as:

$$M = -a^2, \tag{5.40}$$

$$E = \frac{-a^2}{2} - 1. \tag{5.41}$$

Moreover, the constants L and W in the dilaton field are given by:

$$L = a^2, \tag{5.42}$$

$$W = 2a^2. \tag{5.43}$$

From the Einstein field equation ε_{rr}, we find a differential equation for the metric function $R(t)$ that its solutions are given by:

$$R(t) = (\epsilon t + \mu)^{1/a^2}, \tag{5.44}$$

where ϵ and μ are arbitrary constants.

Moreover, by solving the other Einstein equations, we find the metric function $H(t, r, \theta)$ as:

$$H(t, r, \theta) = (R^{a^2+2}(t) + G(r, \theta))^{\frac{2}{a^2+2}} R^{-2}(t), \tag{5.45}$$

where $R(t)$ is given by equation (5.44) and $G(r, \theta)$ is given by:

$$G(r, \theta) = g_+ r^2 \cos\theta + g_-. \tag{5.46}$$

In equation (5.46), g_+ and g_- are arbitrary constants.

We show the behaviour of the metric function $H(t, r, \theta)$ with respect to the coordinates r and θ in figure 5.7, where we consider the specific values for the constants $\epsilon = 1$, $\mu = 2$, $g_+ = 0.5$, $g_- = 15$ and $a = 1$.

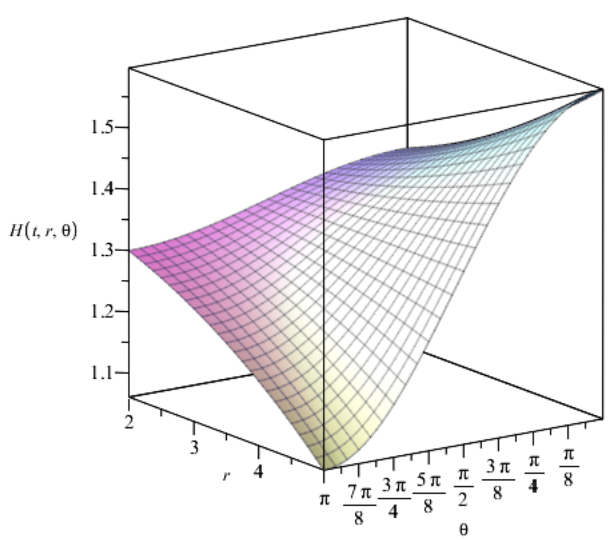

Figure 5.7: The metric function $H(t, r, \theta)$ as a function of the coordinates r and θ, where we set $\epsilon = 1$, $\mu = 2$, $g_+ = 0.5$, $g_- = 15$ and $a = 1$.

From this figure (figure 5.7) we notice that the metric function $H(r,\theta)$ is regular and behaves smoothly in the considered range of the parameters. Moreover, we find that the cosmological constant is given by:

$$\Lambda = \frac{3\epsilon^2(4-a^2)}{2a^4}. \tag{5.47}$$

We notice that the cosmological constant (5.47) can be positive, negative or zero based on the coupling constant a. We verify that all other components of the Einstein and Maxwell equations, as well as the dilaton field equation satisfy with the solutions (5.44)-(5.47). In figure 5.8, we show the behaviour of the c-function versus time, where the cosmological constant is positive. We infer that the $t=$ constant slices for the five-dimensional spacetime (5.37) expands in time.

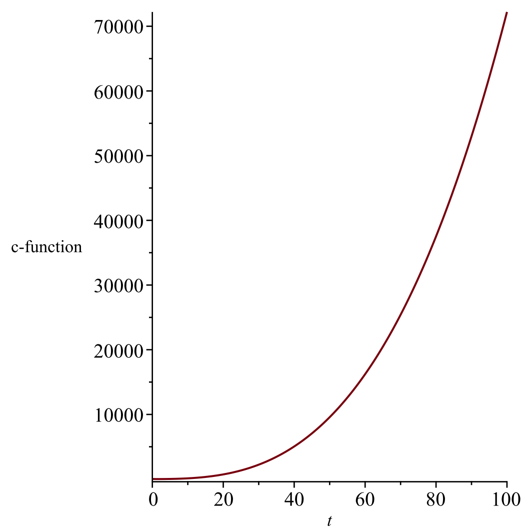

Figure 5.8: The c-function for the five-dimensional spacetime, where the coupling constants are equal.

Moreover, we show the behaviour of the dilaton field $\phi(t,r,\theta)$, in figure 5.9, for two different time slices. The upper slice corresponds to the earlier time. Figures (5.10) and (5.11) show the behaviour of the r and θ components of the electric field. We also note that we cannot find the exact solutions, where the coupling constants a and b are both equal to zero, since the dilaton field (5.39) and the cosmological constant (5.47) diverge. Hence we need another way to find the exact solutions to the theory, where the two coupling constants are equal to zero.

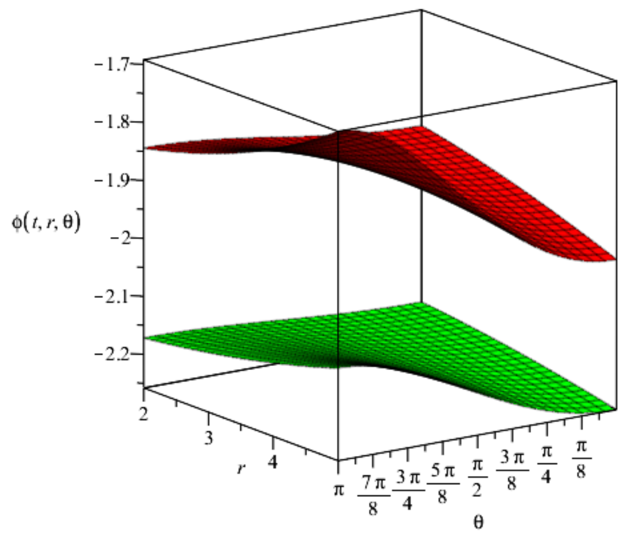

Figure 5.9: The dilaton field $\phi(t, r, \theta)$ for two different time slices $t = 1$ and $t = 2$ (upper and lower hypersurfaces respectively), where $g_+ = 0.5$, $g_- = 15$, $a = 1$, $\epsilon = 1$ and $\mu = 2$.

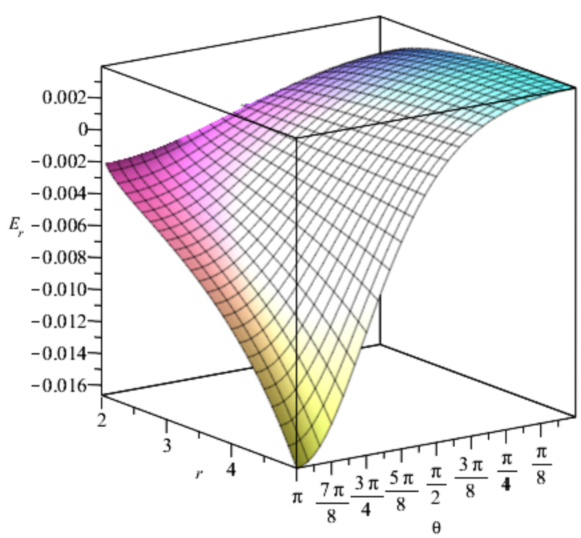

Figure 5.10: The r-component of the electric field as a function of the coordinates r and θ, where $t = 1$. We set the constants as $g_+ = 0.5$, $g_- = 15$, $a = 1$, $\epsilon = 1$, $\mu = 2$, $c = 1$.

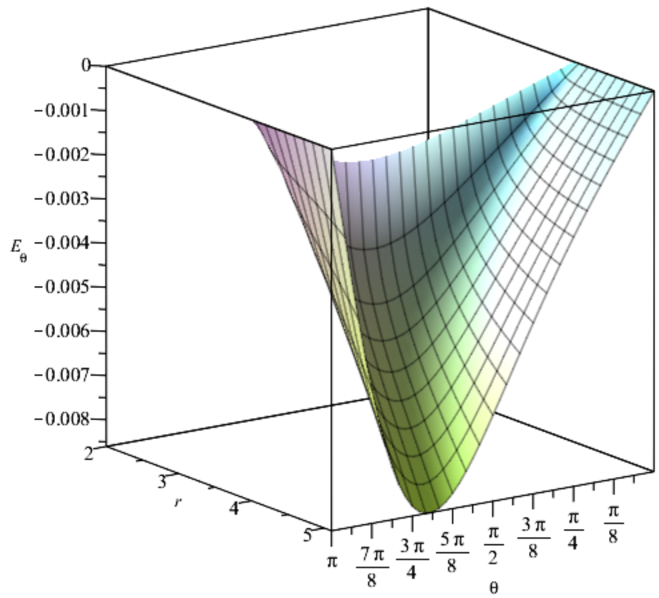

Figure 5.11: The θ-component of the electric field as a function of the coordinates r and θ, where $t = 1$. We set the constants as $g_+ = 0.5$, $g_- = 15$, $a = 1$, $\epsilon = 1$, $\mu = 2$, $c = 1$.

5.3 Exact Solutions Where the Two Coupling Constant are Equal to Zero

As we noticed, the exact solutions, where the coupling constants are both zero cannot be obtained from the solutions in the previous chapter, since the dilaton field (5.39) and the cosmological constant (5.47) diverge in the limit of $a \to 0$. We note that in the limit of $a = b \to 0$ in the action (5.7), the dilaton field would decouple from the Einstein-Maxwell-dilaton theory, and the theory reduces to the Einstein-Maxwell theory with the cosmological constant.

In order to find the exact solutions in the Einstein-Maxwell theory with the cosmological constant, we apply the same ansatz for the five-dimensional metric (5.37) and the following ansatz for the electromagnetic gauge field:

$$A_t(t, r, \theta) = \frac{\alpha}{H(t, r, \theta)}, \tag{5.48}$$

where α is an arbitrary constant. Using the Einstein equations, we find the metric function $R(t)$ as:

$$R(t) = \nu e^{\gamma R_0 t}, \tag{5.49}$$

where ν is a constant, $\gamma = \pm 1$ and $R_0^2 = \Lambda/6$. From the Einstein equation ϵ_{tt}, we find that the constant α in the electromagnetic ansatz is $\alpha = (3/2)^{1/2}$.

Moreover, we find that the metric function $H(t, r, \theta)$ is given by:

$$H(t, r, \theta) = 1 + (f_+ r^2 \cos\theta + f_-) e^{\frac{-\gamma\sqrt{6\Lambda}t}{3}}, \tag{5.50}$$

where f_+ and f_- are arbitrary constants. We verify explicitly that all the other Einstein and Maxwell equations are satisfied.

We show the behaviour of the metric function $H(t, r, \theta)$ in figures 5.12 and 5.13, where we set $\gamma = +1$ and $\gamma = -1$, respectively. We notice that the metric function decreases monotonically with time where $\gamma = +1$, and increases monotonically where $\gamma = -1$.

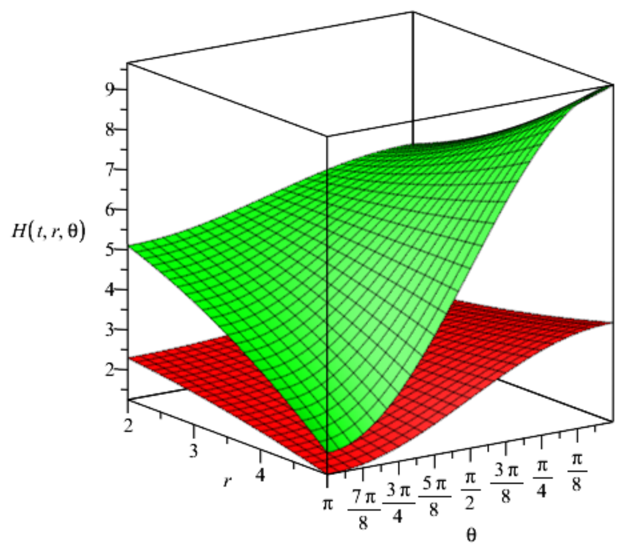

Figure 5.12: The metric function $H(t, r, \theta)$ as a function of the coordinates r and θ for two different time slices $t = 1$ and $t = 2$ (upper and lower surfaces, respectively), where $\gamma = +1$ and the constants are set $f_+ = 0.5$, $f_- = 15$ and $\Lambda = 2$.

We also show the c-function for two different values of $\gamma = +1$ and $\gamma = -1$ in figure 5.14. We notice that the $t =$ constants slices of the spacetime are contracting where $\gamma = +1$, and expanding where $\gamma = -1$.

We also show the components of the electric field in figures 5.15 and 5.16 with respect to the coordinates r and θ.

We found the exact solutions to the Einstein-Maxwell-dilaton theory based on the Bianchi type IX geometry for three different cases, depending on the coupling constants. We showed that each case requires a different set of ansatzes. Our next step is to find the exact solutions to the theory, based on a well-known subspace of the Bianchi type IX geometry. This subspace is the Eguchi-Hanson type II geometry.

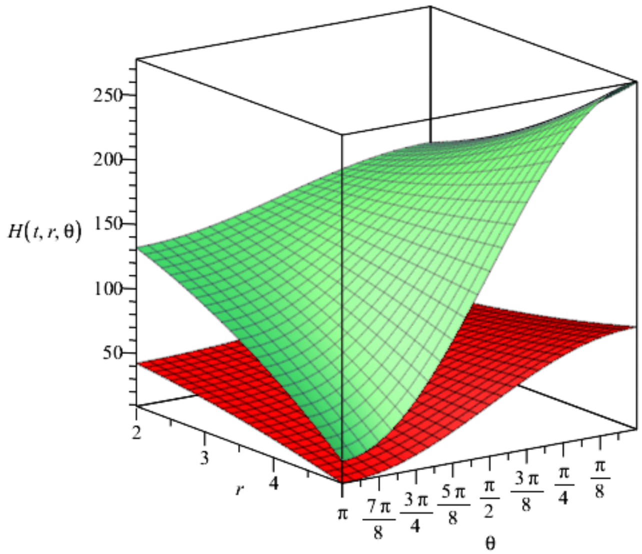

Figure 5.13: The metric function $H(t, r, \theta)$ as a function of the coordinates r and θ for two different time slices $t = 1$ and $t = 2$ (lower and upper surfaces, respectively), where $\gamma = -1$ and the constants are set $f_+ = 0.5$, $f_- = 15$ and $\Lambda = 2$.

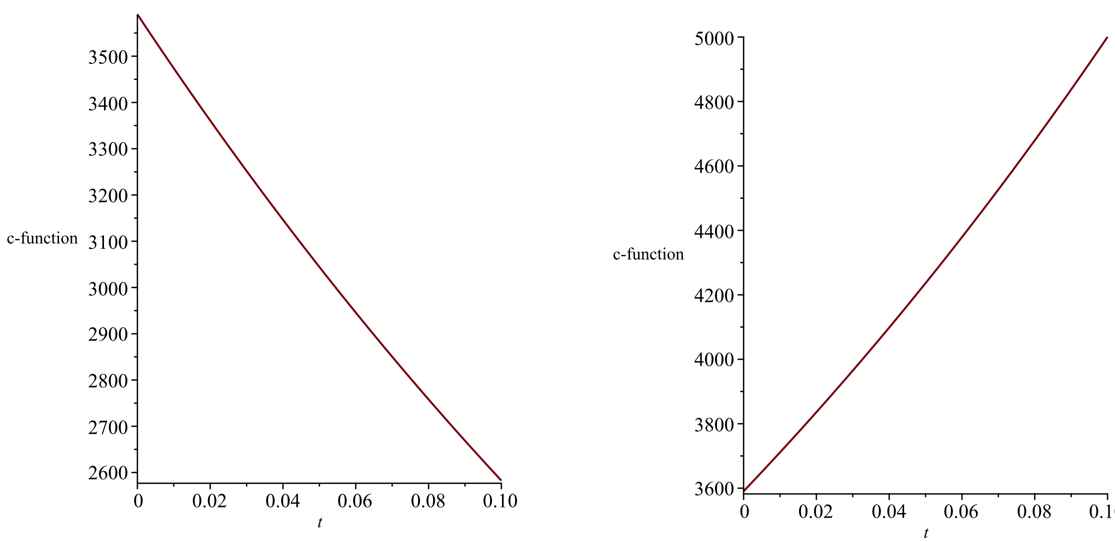

Figure 5.14: The c-functions with $\gamma = +1$ (left) and $\gamma = -1$ (right).

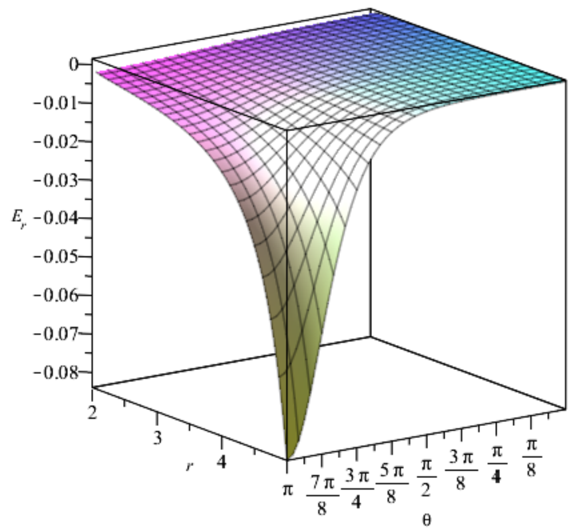

Figure 5.15: The behaviour of the r-component of the electric field with respect to r and θ coordinates for $t = 1$. The constants are set $f_+ = 0.5$, $f_- = 15$, $\Lambda = 2$, $\nu = 3$ and $c = 1$.

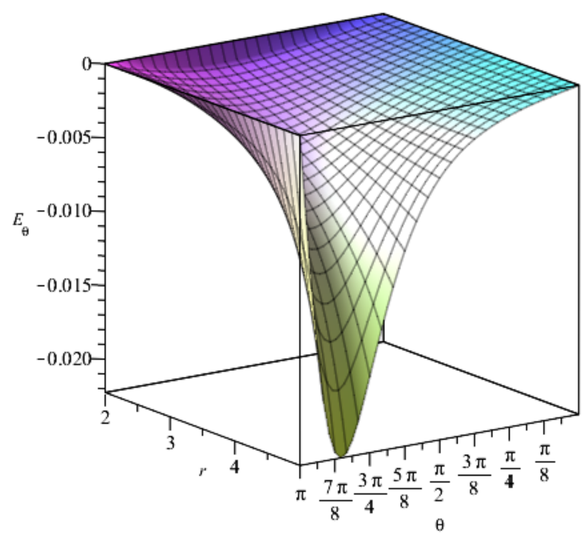

Figure 5.16: The behaviour of the θ-component of the electric field with respect to r and θ coordinates for $t = 1$. The constants are set $f_+ = 0.5$, $f_- = 15$, $\Lambda = 2$, $\nu = 3$ and $c = 1$.

5.4 More General Solutions to the Einstein-Maxwell-dilaton Theory Based on the Eguchi-Hanson Type II Geometry for Non-equal Coupling Constants

The exact solutions to the Einstein-Maxwell-dilaton theory (5.7) are well-known where the transverse four-dimensional space is the Eguchi-Hanson type II geometry [70]. Since the Eguchi-Hanson type II is a subspace of the Bianchi type IX geometry, we can find more general solutions for the Einstein-Maxwell-dilaton theory based on the former geometry. In this section we find a more general class of solutions for the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II geometry.

The metric for the Eguchi-Hanson type II space in four-dimension is given in equation (3.118) [70, 89]. We consider the five-dimensional metric as:

$$ds_5^2 = -\frac{1}{H_{EH}^2(r,\theta)}dt^2 + R^2(t)H_{EH}(r,\theta)ds_{EH.II}^2, \tag{5.51}$$

and use the same ansatzes for the electromagnetic gauge field and the dilaton field, as given in equations (5.12) and (5.14), respectively. We find [70]:

$$H_{EH}(r,\theta) = (1 + \frac{g_+}{r^2 + h^2\cos\theta} + \frac{g_-}{r^2 - h^2\cos\theta})^{\frac{2}{2+a^2}}, \tag{5.52}$$

where $g_\pm$ are arbitrary constants. We note that (5.52) is not a solution for the Einstein-Maxwell-dilaton theory based on the Bianchi type IX space, unless the constant k in the Bianchi metric (3.44) is equal to one. Moreover, due to the nonlinearity of the field equations, the linear summation of (5.28) and (5.52) is not a more general solution. However, we find a general solution for the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II space, which is given by the metric function:

$$\mathcal{H}_{EH}(r,\theta) = (j_+r^2\cos\theta + j_- + \frac{g_+}{r^2 + 4c^2\cos\theta} + \frac{g_-}{r^2 - 4c^2\cos\theta})^{\frac{2}{2+a^2}}, \tag{5.53}$$

where $g_\pm$, $j_\pm$ and c are constants, and a is the coupling constant. The behaviour of the metric function $\mathcal{H}_{EH}(r,\theta)$ with respect to the coordinates r and θ is shown in figure 5.17. We notice from this figure that the metric function $H(r,\theta)$ behaves smoothly in the considered range of the parameters.

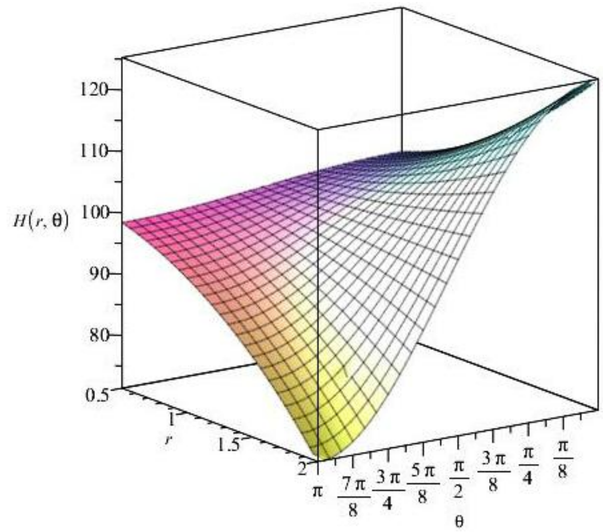

Figure 5.17: The behaviour of the metric function $\mathcal{H}_{EH}(r,\theta)$ with respect to the coordinates r and θ, where we set the constants as $j_+ = 100$, $j_- = 1000$, $g_+0.01$, $g_- = 0.02$, $a = 1$ and $c = 1$.

The other metric function $R(t)$ and the cosmological constant, are still given by (5.30) and (5.31). We also get the same constrain on the coupling constants as (5.32). Considering these results, we find that the Ricci scalar and the Kretschmann invariant for the metric (5.51) diverge at $\mathcal{H}_{EH}(r,\theta) = 0$ and $R(t) = 0$:

$$D_R = -2\mathcal{H}^3 R^2 r^6 \sin\theta (16c^4 - r^4)^3, \tag{5.54}$$

$$D_K = -4\mathcal{H}^6 R^4 r^{12} \sin^5\theta (16c^4 - r^4)^7, \tag{5.55}$$

where D_R and D_K are the denominator of the Ricci scalar and the Kretschmann invariant, respectively. By considering $\eta \geq 0$ and $\vartheta > 0$ in (5.30), we can remove the singularity at $R(t) = 0$.

5.5 More General Solutions where the Non-zero Coupling Constants are Equal

We consider the five-dimensional metric:

$$ds_5^2 = -\frac{1}{H_{EH}^2(t,r,\theta)}dt^2 + R^2(t)H_{EH}(t,r,\theta)ds^2_{EH.II}, \tag{5.56}$$

where the two coupling constants are equal. We also consider the same ansatzes for the electromagnetic field and the dilaton field, which are given in (5.38) and (5.39). The metric function $H(t,r,\theta)$ is obtained in [70] and is given by:

$$H_{EH}(t,r,\theta) = (R^{a^2+2}(t) + K(r,\theta))^{\frac{2}{a^2+2}} R^{-2}(t). \tag{5.57}$$

In (5.57), the function $R(t)$ is given by:

$$R(t) = (\epsilon t + \mu)^{1/a^2}, \tag{5.58}$$

and the function $K(r,\theta)$ is as below:

$$K(r,\theta) = 1 + \frac{k_+}{r^2 + h^2\cos\theta} + \frac{k_-}{r^2 - h^2\cos\theta}, \tag{5.59}$$

where ϵ, μ and $k_\pm$ are arbitrary constants [70]. We again note that due to the nonlinearity of the field equations, the linear summation of (5.45) and (5.57) is not a more general solution. However, we find a general solution for the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II space, which is given by the metric function:

$$\mathcal{H}(t,r,\theta) = \frac{1}{(R(t))^2}\left((R(t))^{a^2+2} + K(r,\theta) + G(r,\theta)\right)^{\frac{2}{2+a^2}}, \tag{5.60}$$

where $G(r,\theta)$ is given in equation (5.46). We also find that the cosmological constant is still given by:

$$\Lambda = \frac{3\epsilon^2(4-a^2)}{2a^4}. \tag{5.61}$$

The behaviour of the metric function $\mathcal{H}(t,r,\theta)$ (5.60) is shown in figure 5.18. We notice from this figure that the metric function $H(r,\theta)$ behaves smoothly in the considered range of the parameters.

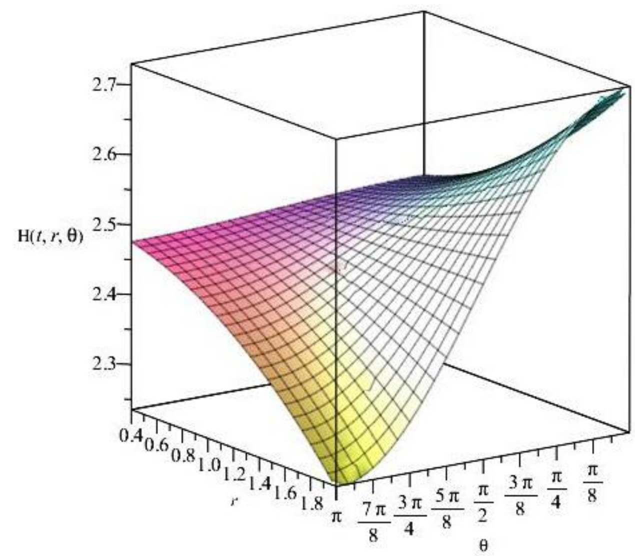

Figure 5.18: The behaviour of the metric function $\mathcal{H}(t,r,\theta)$ with respect to the coordinates r and θ at $t=5$, where the constants are set to be $\epsilon=1$, $\mu=2$, $g_+=50$, $g_-=1000$, $k_+=0.02$, $k_-=0.03$, $a=1$ and $c=1$.

5.6 More General Solutions where the Coupling Constants are Zero

We consider the five-dimensional metric (5.56) along with the electromagnetic field (5.48), where the metric functions $H_{EH}(t,r,\theta)$ is given by [70]:

$$H_{EH}(t,r,\theta) = 1 + \exp\left(\frac{-\gamma\sqrt{6\Lambda}t}{3}\right)\left\{\frac{a_+}{r^2+4c^2\cos\theta} + \frac{a_-}{r^2-4c^2\cos\theta}\right\}. \tag{5.62}$$

In (5.62), $a_\pm$ are two constants and $\gamma = \pm 1$. The other metric function $R(t)$ is given as:

$$R(t) = \nu e^{\gamma R_0 t}, \tag{5.63}$$

where $R_0^2 = \Lambda/6$, ν is a constant and $\gamma = \pm 1$.

We again note that due to the nonlinearity of the field equations, the linear summation of (5.50) and (5.62) is not a more general solution. However, we find a general solution for the Einstein-Maxwell-dilaton theory based on the Eguchi-Hanson type II space, which is given by the metric function,

$$\mathcal{H}(t,r,\theta) = 1 + \exp\left(\frac{-\gamma\sqrt{6\Lambda}t}{3}\right)\left\{\frac{a_+}{r^2+4c^2\cos\theta} + \frac{a_-}{r^2-4c^2\cos\theta}, + j_+ r^2\cos\theta + j_-\right\} \tag{5.64}$$

where $j_\pm$ are constants. We note that $\gamma = +1$, leads the metric function $\mathcal{H}(t,r,\theta)$ decreases monotonically in time, while $\gamma = -1$ leads the metric function $\mathcal{H}(t,r,\theta)$ increases monotonically in time. As an example, we show the behaviour of the metric function $\mathcal{H}(t,r,\theta)$ for $\gamma = 1$ at $t = 1$ in figure 5.19.

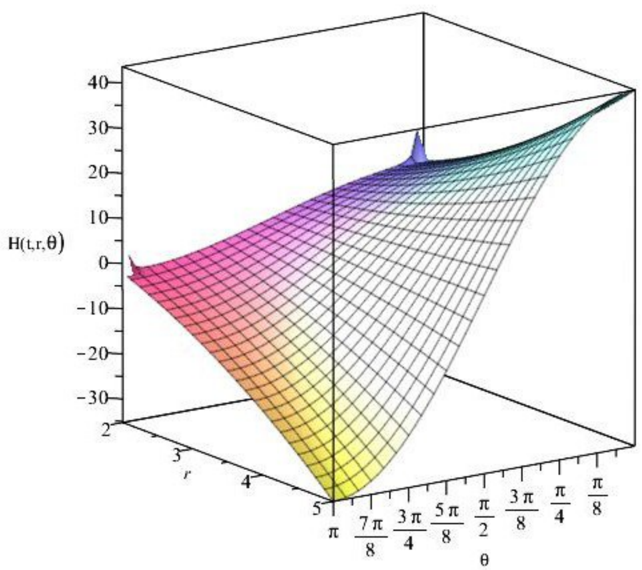

Figure 5.19: The behaviour of the metric function $\mathcal{H}(t,r,\theta)$ with respect to the coordinates r and θ at $t = 1$, where the constants are set to be $\gamma = 1$, $\Lambda = 2$, $j_+ = 5$, $j_- = 10$, $a_+ = 0.1$, $a_- = 0.2$ and $c = 1$.

5.7 Uplifting to Higher Dimensions

We first consider the uplifting of the solutions to the five-dimensional Einstein-Maxwell-dilaton theory to higher than five-dimensional theories where the coupling constants are not equal. Uplifting the solutions of the five-dimensional Einstein-Maxwell-dilaton theory to the Einstein-Maxwell theory in higher than five dimensions ($5 + \mathcal{D}$ dimensions) is possible only if $\mathcal{D}$ satisfies [70, 5, 90]:

$$\mathcal{D} = \frac{3a^2}{1-a^2}, \tag{5.65}$$

and the coupling constants a and b are equal. The latter condition is in contrast to the constrain (5.32), hence we cannot uplift the solutions to the Einstein-Maxwell-dilaton theory to higher than five-dimensional Einstein-Maxwell theory. The other uplifting process is to uplift the solutions to the five-dimensional Einstein-Maxwell-dilaton theory to the solutions of the six-dimensional Einstein gravity with the cosmological constant. The uplifting works only where the coupling constants are $a = \pm 2$ and $b = \pm\frac{1}{2}$ [91]. So, the coupling constants satisfy $ab = 1$, which is in contrast to the equation (5.32). So, we conclude that the solutions to the Einstein-Maxwell-dilaton theory where the coupling constants are not equal, cannot be uplifted to those of the Einstein-Maxwell theory or the Einstein gravity in higher dimensions.

We consider now the uplifting of the solutions to the five-dimensional Einstein-Maxwell-dilaton theory to higher than five-dimensional theories, where the non-zero coupling constants are equal.

The uplifting of the solutions to the five-dimensional Einstein-Maxwell-dilaton theory to those of higher $(5+\mathcal{D})$ dimensional Einstein-Maxwell theory with a cosmological constant is possible, only if equation (5.65) holds. To have $\mathcal{D} \geq 1$, we find the coupling constant a satisfies:

$$\frac{1}{2} \leq a < 1. \tag{5.66}$$

The range of the coupling constant a as given by (5.66), is in contrast to the condition (5.29) for the coupling constant. Therefore, the uplifting of solutions to the Einstein-Maxwell-dilaton theory to a higher dimensional Einstein-Maxwell theory with a cosmological constant is not possible.

Although the solutions that we found for the electromagnetic field, dilaton field and the spacetime metric cannot be uplifted to a known higher dimensional solution with a cosmological constant, we consider another approach based on the reference [90]. Consider the following action for the Einstein gravity with a cosmological constant Λ_D in D-dimensions, where $D = p + q + 1$ and $\mathcal{B}_{[q+1]}$ is a $q+1$-potential [70]:

$$S_D = \int d^D x\sqrt{-g}(R - \frac{1}{2(q+2)!}\mathcal{F}^2_{[q+2]} + 2\Lambda_D), \tag{5.67}$$

where R is the Ricci scalar for the D-dimensional spacetime and $\mathcal{F}_{[q+2]}$ is the $q+2$-field strength form that has the following relation with the $q+1$-potential $\mathcal{B}_{[q+1]}$:

$$\mathcal{F}_{[q+2]} = d\mathcal{B}_{[q+1]}, \tag{5.68}$$

where $d\mathcal{B}_{[q+1]}$ is the exterior derivation of the $\mathcal{B}_{[q+1]}$ potential. We consider the dimensional reduction from D-dimensions to $p+1$-dimensions on an internal curved q-dimensional space, where we show the line element of the internal q-dimensional space by $d\mathcal{K}_q^2$ [70]. We consider the following ansatzes for the D-dimensional metric and the $q+1$-potential $\mathcal{B}_{[q+1]}$:

$$ds_D^2 = e^{-\delta\phi'} ds^2_{p+1} + e^{\phi'(\frac{2}{\delta(p-1)} - \delta)} d\mathcal{K}_q^2, \tag{5.69}$$

$$\mathcal{B}_{[q+1]} = \mathcal{A}_{[1]} \wedge d\mathcal{K}_q. \tag{5.70}$$

The considered ansatzes (5.69) and (5.70) yields to the Einstein-Maxwell-dilaton theory in $p+1$ dimensions with a potential [70]:

$$S_{p+1} = \int d^{p+1}x(R' - \frac{1}{2}(\nabla\phi')^2 - \frac{1}{4}e^{\gamma\phi'}\mathcal{F}^2_{[2]} + 2\Lambda_D e^{-\delta\phi'} + 2\Lambda' e^{-\frac{2}{\delta(p-1)}\phi'}), \tag{5.71}$$

where R' is the Ricci scalar for the $p+1$-dimensional spacetime and $\Lambda' = R''/2$, where R'' represents the Ricci scalar of the internal space. Moreover, in this action (5.71) δ and γ are the dilaton coupling constants with the following relations [90]:

$$\delta = (\frac{2q}{(p-1)(p+q-1)})^{1/2}, \tag{5.72}$$

$$\gamma = \delta(2-p). \tag{5.73}$$

Comparing the action in equation (5.71) and (5.7), we note that by redefining the dilaton field and considering the following relations for the coupling constants, our results for the five-dimensional metric, electromagnetic field and dilaton field can be uplifted to a higher dimensional theory in the absence of the cosmological constant $\Lambda_D = 0$:

$$\phi' = 2\sqrt{\frac{2}{3}}\phi, \tag{5.74}$$

$$\delta = -\frac{3}{4}\sqrt{\frac{3}{2}}\frac{1}{(p-1)b}, \tag{5.75}$$

$$\gamma = -\frac{2}{3}\sqrt{\frac{3}{2}}a. \tag{5.76}$$

According to the relation that we found for the coupling constants $ab = -2$, we find that $p = 4$. Moreover, for having the exact same action as (5.7), we consider:

$$\mathcal{A}_{[1]} = 2A_t dt, \tag{5.77}$$

$$2\Lambda' = -\Lambda. \tag{5.78}$$

Hence, we can uplift our solutions to a higher dimensional theory without a cosmological constant.

5.8 Janis-Newman Method

The rotating metrics are of great interest, as they have many applications and interesting structures such as in cosmology. The solutions to the rotational metrics are especially important in rotating black holes, as they lead to a better understanding and development of quantum gravity. Hence, having a method to generate rotating metrics from the static ones are highly important. As the Einstein equations are highly non-linear, finding the exact solutions to them is not a simple task. There are several off-shell methods that although they do not necessarily preserve the equations of motion, they provide useful ansatzes and insights to the theory [92]. The Janis-Newman method is one of these off-shells methods. We study this method briefly, as for our future goal we would like to generate a rotational metric from the static metrics that we used in our research for the Einstein-Maxwell-dilaton theory, and study the black hole solutions to this theory.

The Janis-Newman method is a powerful tool for generating rotational solutions from the static ones. By using a complex coordinate transformation, Janis and Newman found the derivation of the Kerr metric [93]. Several types of research started to use this method for discovering new axisymmetric solutions to Einstein field equations, with or without the coupling to the Maxwell or dilaton field, and also in different dimensions [94, 95, 96].

We study the generalized Janis-Newman method from the original work that is used to generate the Kerr-Newman metric, which indicates a rotational black hole from the Reinssner-Nordstorm solution. Consider the following static metric [97]:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{5.79}$$

where $f(r)$ is a function of r and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We define the null direction by the following transformation:

$$du = dt - f^{-1}dr. \tag{5.80}$$

Hence, the metric (5.79) becomes:

$$ds^2 = -fdu^2 - 2dudr + r^2 d\Omega^2. \tag{5.81}$$

Janis and Newman used the Newman-Penrose null tetrads formalism, which is [98]:

$$Z_a^\mu = \{l^\mu, n^\mu, m^\mu, \overline{m}^\mu\}, \tag{5.82}$$

where m^μ and $\overline{m}^\mu$ are complex conjugated to each other. The inverse metric takes the form:

$$g^{\mu\nu} = \eta^{ab} Z_a^\mu Z_b^\nu, \tag{5.83}$$

where η^{ab} is given by [92]:

$$\eta_{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5.84}$$

Therefore, the metric $g^{\mu\nu}$ in (5.83) reads:

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \overline{m}^\nu + m^\nu \overline{m}^\mu. \tag{5.85}$$

The tetrads expressions (5.82) are as below [98]:

$$l^\mu = \delta_r^\mu, \tag{5.86}$$

$$n^\mu = \delta_u^\mu - \frac{f}{2}\delta_r^\mu, \tag{5.87}$$

$$m^\mu = \frac{1}{\sqrt{2}r}(\delta_\theta^\mu + \frac{i}{\sin\theta}\delta_\phi^\mu). \tag{5.88}$$

The next step is the complexification of the coordinates r and u. We assume that the coordinates r and u can take complex values if the following restrictions are satisfied [99]:

- m^μ and $\overline{m}^\mu$ must be complex conjugated to each other,

- The tetrads l^μ and n^μ must still be real.

-One should recover the previous basis for $r \in \mathbf{R}$.

As a result for the above conditions, the function $f(r)$ has to be replaced by a new function $\tilde{f}(r, \overline{r}) \in \mathbf{R}$ in a way that $\tilde{f}(r, r) = f(r)$. It is worth noting that one needs to always check to see whether, after this complexification, the solutions still satisfy the Einstein equations or not.

Consider the following complex change of the coordinates:

$$u = u' + ia\cos\theta, \tag{5.89}$$

$$r = r' - ia\cos\theta, \tag{5.90}$$

$$\theta' = \theta, \tag{5.91}$$

$$\phi' = \phi, \tag{5.92}$$

where a is a real parameter (that will be interpreted as the angular momentum per unit of mass [92]). The tetrads become:

$$l'^\mu = \delta^\mu_r, \tag{5.93}$$

$$n'^\mu = \delta^\mu_u - \frac{f}{2}\delta^\mu_r, \tag{5.94}$$

$$m'^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi - ia\sin\theta(\delta^\mu_u - \delta^\mu_r)). \tag{5.95}$$

Hence, we can rebuild the metric tensor $g'^{\mu\nu}$ from the new tetrads and consequently $g'_{\mu\nu}$. Applying this method to the Reissner-Nordstrom yields to the Kerr-Newman rotating solutions [97].

In the Janis-Newman approach it is usually complicated to invert twice the metric and find out the right tetrad basis. Therefore, an alternative way was proposed by Giampieri. We study this method briefly.

Similar to the Janis-Newman method, we define the null coordinate u, which makes the metric (5.79) to become as in equation (5.81). We let the coordinates r and u to take complex values and define the following coordinate transformations [99]:

$$u = u' + ia\cos\eta, \tag{5.96}$$

$$r = r' - ia\cos\eta, \tag{5.97}$$

where the other coordinates are $\theta' = \theta$ and $\phi' = \phi$, and η is a new angle. Hence:

$$du = du' - ia\sin\eta d\eta, \tag{5.98}$$

$$dr = dr' + ia\sin\eta d\eta. \tag{5.99}$$

Omitting the primes, the metric becomes:

$$ds'2 = \tilde{f}(du - ia\sin\eta d\eta)^2 - 2(du - ia\sin\eta d\eta)(dr + ia\sin\eta d\eta) + \rho^2 d\Omega^2, \tag{5.100}$$

where ρ is given by:

$$\rho^2 = r^2 + a^2 \cos^2 \theta. \tag{5.101}$$

The metric in equation (5.100) has to be real. Giampieri proposed the following ansatz for the new defined angle η so that the metric (5.100) reduces to the result from the original formulation [92]:

$$id\eta = \sin \eta d\phi, \tag{5.102}$$

followed by

$$\eta = \theta. \tag{5.103}$$

Although this step is ad hoc, it leads to the correct solution [98].

Therefore, the metric (5.100) reads:

$$ds^2 = -\tilde{f}(du - a \sin^2 \theta d\phi)^2 - 2(du - a \sin^2 \theta d\phi)(dr + a \sin^2 \theta d\phi) + \rho^2 d\Omega^2. \tag{5.104}$$

By redefining the coordinates as below:

$$du = dt' - g(r)dr, \tag{5.105}$$

$$d\phi = d\phi' - h(r)dr, \tag{5.106}$$

We can go to the Boyer–Lindquist coordinate if we impose the constraint $g_{tr} = g_{r\phi'} = 0$ [99]. These condition leads to the following result:

$$g = \frac{r^2 + a^2}{\Delta}, \tag{5.107}$$

$$h = \frac{a}{\Delta}, \tag{5.108}$$

where $\Delta = \tilde{f}\rho^2 + a^2 \sin^2 \theta$. Finally, the metric (5.104) becomes:

$$ds^2 = -\tilde{f}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2} \sin^2 \theta d\phi^2 + 2a(\tilde{f} - 1) \sin^2 \theta dt d\phi, \tag{5.109}$$

where $\frac{\Sigma^2}{\rho^2} = r^2 + a^2$.

As an example, we find the Kerr-Newman solutions by starting from the Reissner-Nordstrom metric. As we discussed in (2.92), the initial metric is:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{5.110}$$

where the function $f(r)$ is given as:

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \tag{5.111}$$

where q is the electric charge and m is the mass.

By complexifying the function $f(r)$, we get [99]:

$$\tilde{f}(r) = 1 + \frac{q^2 - 2mr}{\rho^2}, \tag{5.112}$$

where ρ is given in (5.101). By substituting $\tilde{f}(r)$ in (5.109), we find the Kerr-Newman metric:

$$ds^2 = -\tilde{f}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta d\phi^2 + 2a(\tilde{f}-1)\sin^2\theta dt d\phi. \tag{5.113}$$

In this metric (5.113), Δ and Σ are given by [98]:

$$\frac{\Sigma^2}{\rho^2} = r^2 + a^2 - \frac{q^2 - 2mr}{\rho^2}a^2\sin^2\theta, \tag{5.114}$$

and

$$\Delta = r^2 - 2mr + a^2 + q^2. \tag{5.115}$$

Moreover, the electromagnetic gauge field is given by $A = \frac{q}{r}dt$ in Reissner-Nordstrom solution. By rewriting the gauge field in terms of the new coordinates u and r, we find:

$$A = \frac{q}{r}(du + f^{-1}dr). \tag{5.116}$$

By applying a gauge transformation, the electromagnetic gauge field (5.116) can be rewritten as:

$$A = \frac{q}{r}du, \tag{5.117}$$

where we reach to the final result for the Kerr-Newman gauge field as:

$$A = \frac{qr}{\rho^2}(dt - a\sin^2\theta d\phi). \tag{5.118}$$

Our goal is to find a similar transformation to apply to our results to the Einstein-Maxwell-dilaton theory, in order to construct rotational solutions. We leave this research for our future work.

6 Conclusions

We found new classes of exact solutions to the Einstein-Maxwell-dilaton theory in the presence of the cosmological constant. In this theory, the dilaton field is coupled to both the electromagnetic field and the dilaton field with two different coupling constants via an exponential term. These solutions are non-stationary and almost conformally regular everywhere. We considered the most general case for the coupling constants where they are non-zero and not equal to each other. We considered the four-dimensional Bianchi type IX geometry, as the base space for five-dimensional solutions. This metric is an important geometry in different theories such as string theory and M-theory. We considered ansatzes for the five-dimensional metric, electromagnetic field and the dilaton field. We showed that all of our considered ansatzes satisfy the Einstein, Maxwell and dilaton field equations. In the ansatz for the five-dimensional spacetime metric, we considered two metric functions in a way that one of them depends on two spatial coordinates and the other depends on the time coordinate. By solving the Einstein and Maxwell equations of motion, we found the explicit form of these metric functions and also obtained an important relation between the coupling constants. This relation connects the strength of the interaction of the dilaton-Maxwell field and the dilaton-cosmological constant field. Also we found another constraint on one of the coupling constants. The first few values of the coupling constant through this constraint, leads to the well-known theories such as the low energy heterotic string theory. Moreover, we found the cosmological constant in terms of the coupling constants. We showed that the cosmological constant can be positive, negative or zero depending on the choice of the coupling constant. Finding all the constants and functions through the calculation, we showed the behaviour of the metric functions, dilaton field and the electric field in different plots. Furthermore, we studied the uplifting of the found solutions to higher dimensional theories such as Einstein gravity in higher dimensions and Einstein-Maxwell theory with a cosmological constant and we showed that our solutions cannot be found from the compactification of these theories.

Moreover, by considering an ansatz for the five-dimensional spacetime, we generated the c-function and discussed the properties of the spacetime. Furthermore, we showed that the case, where the coupling constants are non-zero and equal to each other can not be obtained from the previous case and requires another set of ansatzes. By considering the new set of ansatzes for the five-dimensional metric, electromagnetic field and dilaton, we found the explicit form of the considered functions and calculated the cosmological constant in terms of the coupling constant. The cosmological constant, in this case, can also be positive, negative or zero. The Einstein-Maxwell-dilaton theory reduces to the simple Einstein-Maxwell theory when the coupling constants are equal to zero. We considered a new set of assumptions for the fields and found the exact solutions to this theory.

We proposed a new set of combined solutions based on the four-dimensional Eguchi-Hanson geometry. This geometry is a sub-space of the four-dimensional Bianchi type IX geometry. Exact cosmological solutions to the Einstein-Maxwell-dilaton theory based on the four-dimensional Eguchi-Hanson metric is found in the article [70]. We combined the general solutions with the well-known cosmological solutions based on the Eguchi-Hanson metric in a non-linear way and showed that these combined solutions satisfy all the field equations. We showed that the linear combination of the solutions is not simply a solution to the theory.

For our future work, we will work on rotational solutions to the Einstein-Maxwell-dilaton theory. As the rotational solutions play an important role in gravitational physics such as in cosmology and black hole physics, we would like to see whether we can generate rotational solutions from the exact static solutions that we found to the Einstein-Maxwell-dilaton theory based on the Bianchi type IX geometry.

7 Appendix

For the case where the coupling constant are non-zero and not equal to each other, the Maxwell field equation M_t gives a partial differential equation for the metric function $H(r,\theta)$, and is given by:

$$\begin{aligned}
M_t &= \frac{1}{4a}(E\alpha R^M H^E(-1024\sin\theta H(\frac{\partial H}{\partial r})c^8k^4a + 1024r\sin\theta(\frac{\partial H}{\partial r})^2c^8k^4a - 256\cos^2\psi\cos\theta \\
&\times (\frac{\partial H}{\partial \theta})c^4k^4r^3Ha - 64r^5\sin\theta(\frac{\partial H}{\partial r})^2c^4k^4a - 256\cos^2 psi(\frac{\partial H}{\partial \theta})^2c^4\sin\theta r^3a - 64E \\
&\times (\frac{\partial H}{\partial r})^2\sin\theta c^4r^5a - 64r^4\sin\theta H(\frac{\partial H}{\partial r})c^4a - 36a(\frac{\partial H}{\partial r})^2\sin\theta c^4r^5L + 256\cos^2\psi(\frac{\partial H}{\partial \theta})^2 \\
&\times \sin\theta ac^4k^4r^3 + 256\cos^2\psi H(\frac{\partial H}{\partial \theta})\cos\theta ac^4r^3 - 64r^5\sin\theta(\frac{\partial H}{\partial \theta})^2 \\
&\times c^4a + 12r^8\sin\theta H(\frac{\partial H}{\partial r})a + 16a(\frac{\partial H}{\partial \theta})^2\sin\theta r^7L + 16\cos\theta H(\frac{\partial H}{\partial \theta})r^7a \\
&+ 16E(\frac{\partial H}{\partial \theta})^2\sin\theta r^7a + 4E(\frac{\partial H}{\partial r})^2\sin\theta r^9a + 4a(\frac{\partial H}{\partial r})^2\sin\theta r^9L - 256E\cos^2\psi(\frac{\partial H}{\partial \theta})^2 \\
&\times c^4\sin\theta r^3a - 256a\cos^2\psi(\frac{\partial H}{\partial \theta})^2\sin\theta c^4r^3L + 4r^9\sin\theta H(\frac{\partial^2 H}{\partial r^2})a \\
&+ 16\sin\theta H(\frac{\partial^2 H}{\partial \theta^2})r^7a - 256\cos^2\psi(\frac{\partial^2 H}{\partial \theta^2})c^4\sin\theta r^3Ha - 256H(\frac{\partial^2 H}{\partial \theta^2})c^4k^4r^3a \\
&+ 1024r\sin\theta H(\frac{\partial^2 H}{\partial r^2})c^8k^4a - 64r^5\sin\theta H(\frac{\partial^2 H}{\partial r^2})c^4k^4a + 256a\cos^2\psi(\frac{\partial H}{\partial \theta})^2 \\
&\times \sin\theta Lc^4k^4r^4 + 256\cos^2\psi(\frac{\partial H}{\partial \theta})^2\sin\theta Eac^4k^4r^3 - 256a(\frac{\partial H}{\partial \theta})^2\sin\theta c^4k^4r^3L \\
&- 64a(\frac{\partial H}{\partial r})^2\sin\theta c^4k^4r^5L + 1024a(\frac{\partial H}{\partial r})^2\sin\theta c^8k^4rL - 64r^4\sin\theta H(\frac{\partial H}{\partial r})c^4k^4a \\
&- 64r^5\sin\theta H(\frac{\partial^2 H}{\partial r^2})c^4a - 256E(\frac{\partial H}{\partial \theta})^2\sin\theta c^4k^4r^3a + 1024E(\frac{\partial H}{\partial r})^2\sin\theta c^8k^4ra \\
&- 64E(\frac{\partial H}{\partial r})^2\sin\theta c^4k^4r^5a + 256\cos^2\psi H(\frac{\partial^2 H}{\partial \theta^2})\sin\theta ac^4k^4r^3 - 256\sin\theta(\frac{\partial H}{\partial \theta})^2 \\
&\times c^4k^4r^3a - 256\cos\theta r^3H(\frac{\partial H}{\partial \theta})c^4a + 16\sin\theta(\frac{\partial H}{\partial \theta})^2r^7a + 4r^9\sin\theta(\frac{\partial H}{\partial \theta})^2a. \qquad (7.1)
\end{aligned}$$

We find that solutions to equation (7.1) are given by:

$$H(r,\theta) = (j_+r^2\cos\theta + j_-)^{\frac{2}{a^2+2}}, \qquad (7.2)$$

where j_+ and j_- are constants.

References

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